2020-21 Onwards (MR-20)	**MALLA REDDY ENGINEERING COLLEGE (Autonomous)**	**B.Tech. I Semester**		
Code:A0B01	**Linear Algebra and Numerical Methods**	**L**	**T**	**P**
Credits: 4	**(Common for CSE & IT)**	**3**	**1**	**-**

Prerequisites: Matrices, Differentiation and Integration.

Course Objectives:

1. To learn types of matrices, Concept of rank of a matrix and applying the concept of rank to know the consistency of linear equations and to find all possible solutions, if exist.
2. To learn concept of Eigen values and Eigen vectors of a matrix, diagonalization of a matrix, Cayley Hamilton theorem and reduce a quadratic form into a canonical form through a linear transformation.
3. To learn various methods to find roots of an equation.
4. To learn Concept of finite differences and to estimate the value for the given data using interpolation.
5. To learn Solving ordinary differential equations and evaluation of integrals using numerical techniques.

MODULE I: Matrix algebra **[12 Periods]**

Vector Space, basis, linear dependence and independence (Only Definitions)

Matrices: Types of Matrices, Symmetric; Hermitian; Skew-symmetric; Skew- Hermitian; orthogonal matrices; Unitary Matrices; Rank of a matrix by Echelon form and Normal form, Inverse of Non-singular matrices by Gauss-Jordan method; solving system of Homogeneous and Non-Homogeneous linear equations, LU – Decomposition Method.

MODULE II: Eigen Values and Eigen Vectors **[12 Periods]**

Eigen values , Eigen vectors and their properties; Diagonalization of a matrix; Cayley-Hamilton Theorem (without proof); Finding inverse and power of a matrix by Cayley-Hamilton Theorem; Singular Value Decomposition.

Quadratic forms: Nature, rank, index and signature of the Quadratic Form, Linear Transformation and Orthogonal Transformation, Reduction of Quadratic form to canonical forms by Orthogonal Transformation Method.

MODULE III: Algebraic &Transcendental equations **[12 Periods]**

(A) Solution of Algebraic and Transcendental Equations: Introduction-Errors, types of errors. Bisection Method, Method of False Position, Newton-Raphson Method.

(B) The Iteration Method ,Ramanujan's method to find smallest root of Equation. Jacobi's Iteration method. Gauss seidel Iteration method.

MODULE IV: Interpolation **[12 Periods]**

Introduction- Errors in Polynomial Interpolation – Finite differences- Forward Differences-Backward differences-Central differences - Symbolic relations and separation of symbols. Differences of a polynomial-Newton's formulae for interpolation; Central difference interpolation Formulae – Gauss Central Difference Formulae ; Interpolation with unevenly spaced points-Lagrange's Interpolation formula.

MODULE V: Numerical solution of Ordinary Differential Equations and Numerical Integration **[12 Periods]**

Numerical solution of Ordinary Differential Equations : Introduction-Solution of Ordinary Differential Equation by Taylor's series method - Picard's Method of successive Approximations - Euler's Method-Modified Euler's Method – Runge-Kutta Methods.

Numerical Integration: Trapezoidal Rule, Simpson's 1/3rd Rule, Simpson's 3/8 Rule.

TEXT BOOKS

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010.
2. Erwin kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006.
3. D. Poole, Linear Algebra: A Modern Introduction, 2nd Edition, Brooks/Cole, 2005.
4. M . K Jain, S R K Iyengar, R.K Jain, Numerical Methods for Scientific and Engineering Computation, New age International publishers.
5. S.S.Sastry, Introductory Methods of Numerical Analysis,5th Edition,PHI Learning Private Limited.

REFERENCES

1. G.B. Thomas and R.L. Finney, Calculus and Analytic geometry, 9th Edition, Pearson, Reprint,2002.
2. N.P. Bali and Manish Goyal, A text book of Engineering Mathematics, Laxmi Publications,
   Reprint, 2008.
3. V. Krishnamurthy, V.P. Mainra and J.L. Arora, An introduction to Linear Algebra, AffiliatedEast–West press, Reprint 2005.

4. Ramana B.V., Higher Engineering Mathematics, Tata McGraw Hill New Delhi, 11th Reprint,2010.

E – RESOURCES

1. https://www.youtube.com/watch?v=sSjB7ccnM_I (Matrices – System of linear Equations)
2. https://www.youtube.com/watch?v=h5urBuE4Xhg (Eigen values and Eigen vectors)
3. https://www.youtube.com/watch?v=9y_HcckJ96o (Quadratic forms)
4. https://www.youtube.com/watch?v=3j0c_FhOt5U (Bisection Method)
5. https://www.youtube.com/watch?v=6vs-pymcsqk (Regula Falsi Method and Newton Raphson Method)
6. https://www.youtube.com/watch?v=1pJYZX-tgi0 (Interpolation)
7. https://www.youtube.com/watch?v=Atv3IsQsak8&pbjreload=101 (Numerical Solution of ODE)
8. https://www.youtube.com/watch?v=iviiGB5vxLA (Numerical Integration)

NPTEL

1. https://www.youtube.com/watch?v=NEpvTe3pFIk&list=PLLy_2iUCG87BLKl8eISe4fHKdE2_j2B_T&index=5 (Matrices – System of linear Equations)
2. https://www.youtube.com/watch?v=wrSJ5re0TAw (Eigen values and Eigen vectors)
3. https://www.youtube.com/watch?v=yuE86XeGhEA (Quadratic forms)
4. https://www.youtube.com/watch?v=WbmLBRbp0zA (Bisection Method)
5. https://www.youtube.com/watch?v=0K6olBTdcSs (Regula Falsi and Newton Raphson Method)
6. https://www.youtube.com/watch?v=KSFnfUYcxoI (Interpolation)
7. https://www.youtube.com/watch?v=QugqSa3Gl-w&t=2254s (Numerical Solution of ODE)
8. https://www.youtube.com/watch?v=NihKCpjx2_0&list=PLbMVogVj5nJRILpJJO7KrZa8Ttj4_ZAgl
(Numerical Solution of ODE)
9. https://www.youtube.com/watch?v=hizXlwJO1Ck (Numerical Integration)

Course Outcomes:

1. The student will be able to find rank of a matrix and analyze solutions of system of linear equations.
2. The student will be able to find Eigen values and Eigen vectors of a matrix, diagonalization a matrix, verification of Cayley Hamilton theorem and reduce a quadratic form into a canonical form through a linear transformation.
3. The student will be able to find the root of a given equation by various methods.
4. The student will be able to estimate the value for the given data using interpolation.
5. The student will be able to find the numerical solutions for a given ODE's and evaluations of integrals using numerical techniques.

CO- PO Mapping

CO- PO, PSO Mapping (3/2/1 indicates strength of correlation) 3-Strong, 2-Medium, 1-Weak												
COS	**Programme Outcomes(POs)**											
	PO 1	**PO 2**	**PO 3**	**PO 4**	**PO 5**	**PO 6**	**PO 7**	**PO 8**	**PO 9**	**PO 10**	**PO 11**	**PO 12**
CO1	3	2	2	3	3				2			1
CO2	2	2	2	3	2				2			1
CO3	2	2	2	3	2				2			1
CO4	3	2	2	3	3				2			2
CO5	2	2	2	3	3				2			2

MODULE -I

MATRIX ALGEBRA

MATRICES

Matrix :—

An arrangement of mn numbers (real or complex) in a rectangular array having m rows (Horizontal lines) and n columns (vertical lines), the numbers being enclosed by brackets [ ] or ( ) is called an m×n matrix (read as m by n matrix)

Here m×n is called as the order or type of a matrix and each of mn numbers is called as an element of matrix.

Generally matrices are denoted by capital letters A, B, C, ...

and its elements are denoted by small letters a, b, c, ...

An m×n matrix can be expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

It is briefly written as $A = [a_{ij}]_{m\times n}$

where $i = 1, 2, 3 \ldots m$ stands for rows

$j = 1, 2, 3 \ldots n$ stands for columns.

Eg:- $\begin{bmatrix} 7 & -2 & 0 \\ 8 & -3 & 1 \end{bmatrix}$ is a matrix of order 2×3

$\begin{bmatrix} 1 & 8 \\ 3 & 27 \end{bmatrix}$ is a matrix of order 2×2

Types of Matrices :—

Row Matrix :—

A matrix having only one row and any number of columns is said to be a row matrix. It is of order $1 \times n$.

Eg:- $\begin{bmatrix} -1 & 0 & 1 & 2 \end{bmatrix}$ is a row matrix of order 1×4

Column Matrix :—

A matrix having only one column and any number of rows is said to be a column matrix. It is of order $n \times 1$.

Eg:- $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a column matrix of order 4×1

Rectangular Matrix :—

A matrix having rows and columns are not equal is said to be a rectangular matrix. It is of order $m \times n$.

Eg:- $\begin{bmatrix} 1 & 2 & 7 \\ 4 & 5 & 9 \end{bmatrix}$ is a rectangular matrix of order 2×3

Square Matrix :—

A matrix having rows and columns are equal is said to be a square matrix. It is of order $n \times n$ or square matrix of order n.

Eg:- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a square matrix of order 2.

Principal diagonal of a Square matrix :—

In a matrix $A = [a_{ij}]_{n\times n}$, the diagonal which carries from the first row first element to last rows last element is called the principal diagonal of A.

The elements a_{jj} of A for which $i=j$ i.e $a_{11}, a_{22}, a_{33}, \ldots a_{nn}$ are called the elements of the principal diagonal of A.

Trace of a Square matrix :—

Let $A = [a_{ij}]_{n\times n}$

The sum of the elements of the principal diagonal elements is called the Trace of A. and is denoted by tr A.

$$\therefore \ tr A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

Properties :—

If A and B are square matrices of order n and λ is any scalar then

(i) $tr(\lambda A) = \lambda \, tr(A)$

(ii) $tr(A+B) = tr A + tr B$.

(iii) $tr(AB) = tr(BA)$.

Diagonal Matrix :—

A square matrix in which all the elements except in the principal diagonal are zero is called a diagonal matrix.

Eg :- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a square matrix of order 3. is a diagonal matrix of order 3.

Eg:- If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$

Properties :- If A^T and B^T are the transposes of A and B respectively then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$ Where A and B are of the same order.

(iii) $(KA)^T = KA^T$, Where K is a scalar.

(iv) $(AB)^T = B^T A^T$ Where A and B are conformable for multiplication

Symmetric Matrix :-

A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j

Thus the necessary and sufficient condition for a square matrix A to be symmetric is that $A^T = A$

$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix}$ are symmetric matrices of order 3.

Skew Symmetric Matrix :-

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji}$ for every i and j

Thus A is skew symmetric matrix iff $A^T = -A$

Thus all the diagonal elements of a skew symmetric matrix are zero.

Eg: $\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$ are skew symmetric matrices of order 3.

Note :- (i) A is symmetric $\Longrightarrow$ KA is symmetric.

(ii) A is skew symmetric $\Longrightarrow$ KA is skew symmetric.

Properties :-

(i) Inverse of a non singular symmetric matrix A is symmetric.

(ii) If A is a symmetric matrix then adj A is also symmetric.

(iii) If A is a $m \times n$ matrix and B is a $n \times p$ matrix then $(AB)^T = B^T A^T$.

Theorem :- Every square matrix can be expressed as the sum of a symmetric and skew symmetric matrices in one and only way [OR] show that any square matrix $A = B + C$ where B is symmetric and C is skew symmetric matrices.

Proof :- Let A be any square matrix.

We can write $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$A = P + Q$ say.

where $P = \frac{1}{2}(A + A^T)$

$Q = \frac{1}{2}(A - A^T)$

$\because A = \frac{1}{2}A + \frac{1}{2}A$

$A = \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T + \frac{1}{2}A$

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

We have $P = \frac{1}{2}(A + A^T)$

$P^T = \left[\frac{1}{2}(A + A^T)\right]^T$

$= \frac{1}{2}(A + A^T)^T$

$= \frac{1}{2}(A^T + (A^T)^T)$

$P^T = \frac{1}{2}(A^T + A)$

$P^T = P$.

$\therefore$ P is symmetric matrix.

We have $Q = \frac{1}{2}(A - A^T)$

$Q^T = \left[\frac{1}{2}(A - A^T)\right]^T$

$= \frac{1}{2}(A - A^T)^T$

$$Q^T = \frac{1}{2}(A - A^T)^T$$

$$= \frac{1}{2}(A^T - (A^T)^T)$$

$$= \frac{1}{2}(A^T - A)$$

$$= -\frac{1}{2}(A - A^T)$$

$$Q^T = -Q$$

$\therefore$ Q is skew symmetric matrix.

Thus, square matrix = symmetric + skew symmetric.

Thus, A is a sum of symmetric matrix and a skew symmetric matrix.

To Prove that the sum is unique :-

If possible, let $A = R + S$ be another such representation of A

Where R is a symmetric and S is a skew symmetric matrix.

$\therefore \quad R^T = R$ and $S^T = -S$.

Now $\quad A^T = (R+S)^T = R^T + S^T = R - S$

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}(R + S + R - S) = R.$$

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}(R + S - R + S) = S.$$

$\Rightarrow R = P$ and $S = Q$.

Thus, the representation is unique.

Express the matrix $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrices.

Sol:- Given that $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$

We know that symmetric part of matrix A is $P = \frac{1}{2}(A + A^T)$

and skew symmetric part of matrix A is $Q = \frac{1}{2}(A - A^T)$

$$A^T = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix} = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$$

$$P^T = P$$

$\therefore$ P is symmetric.

$$A - A^T = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$Q^T = -Q$$

$\therefore$ Q is skew symmetric.

$$A = P + Q = \frac{1}{2}\begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

Properties :-

(i) The inverse of a non singular symmetric matrix A is symmetric.

(ii) If A and B are symmetric matrices then AB is symmetric if and only if $AB = BA$.

(iii) If A is any matrix then AA^T and A^TA are both symmetric.

(iv) The matrix B^TAB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

(v) All positive Integral powers of a symmetric matrix are symmetric

(vi) Positive odd integral powers of a skew symmetric matrix are skew symmetric where as positive even integral powers are symmetric.

Orthogonal Matrix :—

A square matrix A is called an orthogonal matrix if $AA^T = A^TA = I$.

(1) Determine the values of a, b, c such that $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is an orthogonal matrix.

sol:- Given that $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$

By def. A is an orthogonal $\Rightarrow AA^T = A^TA = I$.

$$AA^T = I \Longrightarrow \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the elements of corresponding positions, we get

$4b^2+c^2 = 1$ — (1) $\quad 2b^2-c^2 = 0$ — (2)

$a^2+b^2+c^2 = 1$ — (3) $\quad a^2-b^2-c^2 = 0$ — (4)

$(1) + (2) \Longrightarrow 6b^2 = 1 \Longrightarrow b = \pm\frac{1}{\sqrt{6}}$

$(3) + (4) \Longrightarrow 2a^2 = 1 \Longrightarrow a = \pm\frac{1}{\sqrt{2}}$

from (2), $c^2 = 2b^2$

$c^2 = 2\frac{1}{6} = \frac{1}{3}$

$c = \pm\frac{1}{\sqrt{3}}$

$\therefore a = \pm\frac{1}{\sqrt{2}},\ b = \pm\frac{1}{\sqrt{6}}\ c = \frac{\pm 1}{\sqrt{3}}$ are required values.

(2) show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \\ 0 & -\cos\theta & \sin\theta \end{bmatrix}$ is an orthogonal.

(3) show that $A = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ is orthogonal.

(4) S/T. $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is orthogonal.

(5) S/T $A = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ \sin\theta\sin\phi & \cos\theta & -\sin\theta\cos\phi \\ -\cos\theta\sin\phi & \sin\theta & \cos\theta\cos\phi \end{bmatrix}$ is orthogonal.

(6) Find a +ve integer "a" such that $\frac{1}{a}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$ is orthogonal.

Properties:-

(i) If A is orthogonal matrix then $|A| = \pm 1$

(ii) The inverse of an orthogonal matrix is orthogonal.

(iii) The transpose of an orthogonal matrix is orthogonal.

(iv) If A, B be orthogonal matrices, AB and BA are also orthogonal.

→ Reduce the matrix $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ to echelon form and find its rank.

Sol:- Let $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}_{4\times4}$

Now we reduce the matrix A into echelon form by applying row operations only.

$R_2 \to R_2 + R_1 \quad R_3 \to R_3 + 2R_1 \quad R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

$R_3 \to 2R_3 - 11R_2 , \quad R_4 \to R_4 + 2R_2$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_4 \to 6R_4 + R_3$

$$A \sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is in echelon form

$\therefore \rho(A) =$ No. of non zero rows of the last equivalent to A $= 4$.

$\therefore \rho(A) = 4$

→ Apply elementary transformations to find the rank of

$$A = \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$$

Sol:- Given that $A = \begin{bmatrix} 1 & -7 & 3 & -3 \\ 7 & 20 & -2 & 25 \\ 5 & -2 & 4 & 7 \end{bmatrix}$

Now we reduce the matrix A into echelon form by applying row operations only.

$R_2 \to R_2 - 7R_1,\quad R_3 \to R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 69 & -23 & 46 \\ 0 & 33 & -11 & 22 \end{bmatrix}$$

$R_2 \to R_2\left(\frac{1}{23}\right) \quad R_3 \to R_3\left(\frac{1}{11}\right)$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 3 & -1 & 2 \\ 0 & 3 & -1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -7 & 3 & -3 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.

$\therefore \rho(A) =$ The no. of non zero rows of the last equivalent of A $= 2$

$\therefore \rho(A) = 2.$

→ Find the constants λ and m suchthat the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & \lambda & m \end{bmatrix}$$ is (i) 3 (ii) 2

Sol:- Given that $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & 2 & \lambda & m \end{bmatrix}$

Now we reduce the matrix A into echelon form by applying row operations only.

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 6R_1$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 10 & \lambda-18 & m-6 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & \lambda-4 & m-6 \end{bmatrix}$$

Which is in echelonform

(i) $\rho(A) = 3$ if $\lambda \neq 4$ or $m \neq 6$

(ii) $\rho(A) = 2$ if $\lambda = 4$ and $m = 6$.

→ For what value of k the matrix $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank 3.

Sol. Given that $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$

$R_2 \to 4R_2 - R_1,\ R_3 \to 4R_3 - kR_1,\ R_4 \to 4R_4 - 9R_1$

$$\sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$$

The given matrix is of order 4×4. If its rank is 3, then we must have $|A| = 0$

$$\Rightarrow 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

$$1\left[(8-4k)3\right] - 1\left[(8-4k)(4k+27)\right] = 0.$$

$$(8-4k)\left[3 - 4k - 27\right] = 0$$

$$(8-4k)(-24-4k) = 0.$$

$\therefore$ $k = 2$ or $k = -6$.

ECHELON FORM.

1 Define Echelon form of a matrix.

2 Find the rank of matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by reducing it into echelon form Ans:- 2.

3 Find the rank of matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ by reducing it into echelon form Ans:- 3

4 Find the value of k so that the rank of the matrix $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$ is three. Ans:- $k = 2$ or $k = -6$.

5 Find the value of k, if the Rank of matrix A is 2 $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

Ans:- $k = -2$

6 Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 1 & 2 \\ -3 & -4 & 5 & 8 \\ 1 & 3 & 10 & 14 \end{bmatrix}$ by reducing it into echelon form Ans:- 2.

7 Find the rank of the matrix. $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ by reducing it into echelon form Ans:- 4.

8 Find the rank of the matrix $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ by reducing it into echelon form. Ans:- 2

Normal form or Canonical form of a matrix :-

If an $m \times n$ matrix can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by using a finite chain of elementary operations. Where I_r is the unit matrix of order r and '0' is the null matrix then the above form is called "The normal form" or "The first canonical form of a matrix". Here r indicates the rank of a matrix.

The various normal forms are I_r, $[I_r \ 0]$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ and $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Working procedure to reduce a matrix to the canonical form :-

Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Step (i) :- If $a_{11} \neq 0$, by using a_{11} position make a_{21} and a_{31} positions as zero. Here we use row operations.

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \end{bmatrix}$$

Step (ii) :- By using a_{11} position make a_{12}, a_{13} and a_{14} positions as zero. Here we use column operations.

$$\sim \begin{bmatrix} a_{11} & 0 & 0 & a \\ 0 & a''_{22} & a''_{23} & a''_{24} \\ 0 & a''_{32} & a''_{33} & a''_{34} \end{bmatrix}$$

Step (iii) :- If $a''_{22} \neq 0$, by using a''_{22} position make a''_{32} position as zero. Here we use row operation.

$$\sim \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a''_{22} & a''_{23} & a''_{24} \\ 0 & 0 & a'''_{33} & a'''_{34} \end{bmatrix}$$

Step (iv) :- By using a''_{22} position make a''_{23} and a''_{24} positions as zero. Here we use column operations.

$$\sim \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a''_{22} & 0 & 0 \\ 0 & 0 & a^{iv}_{33} & a^{iv}_{34} \end{bmatrix}$$

Step (v) :- If $a^{iv}_{33} \neq 0$, by using a^{iv}_{33} position make a^{iv}_{34} position as zero. Here we use column operation.

$$\sim \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a''_{22} & 0 & 0 \\ 0 & 0 & a^{iv}_{33} & 0 \end{bmatrix}$$

Step (vi) :- By using suitable elementary operations make a_{11}, a''_{22} and a^{iv}_{33} positions as one. Now which is of the form

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$A \sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

$$\therefore \quad \rho(A) = 3.$$

→ Find the rank of a matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ by reducing it to canonical form.

Sol: Given that $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

Now we reduce the matrix A into normal form by applying row and column operations.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ $R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$ $C_3 \to C_3 - 3C_1$ $C_4 \to C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$ $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2$ $C_4 \to C_4 - 3C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2(-1)$

$$A \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which is of the form $A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Which is in normal form.

$\therefore \rho(A) = 2.$

→ Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & k \end{bmatrix}$ by reducing it to the canonical form.

Sol:- Given that $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & k \end{bmatrix}$

~~Sol~~ Now we reduce the matrix A into ~~echelon~~ normal form by applying elementary row and column operations.

$R_2 \to R_2 - 4R_1, \; R_3 \to R_3 - 3R_1 \;\; R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & k-3 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1 \quad C_3 \to C_3 + C_1 \quad C_4 \to C_4 - 3C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & k-3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & k-3 \end{bmatrix}$$

$C_3 \rightarrow 7C_3 + 6C_2 \quad C_4 \rightarrow 7C_4 - 11C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -14 & 28 \\ 0 & 0 & 7 & 7k-21 \end{bmatrix}$$

$R_4 \rightarrow 2R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -14 & 28 \\ 0 & 0 & 0 & 14k-14 \end{bmatrix}$$

$C_4 \rightarrow C_4 + 2C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -14 & 0 \\ 0 & 0 & 0 & 14k-14 \end{bmatrix}$$

$R_3 \rightarrow R_3\left(-\frac{1}{7}\right) \quad R_4 \rightarrow R_4\left(-\frac{1}{14}\right)$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 14k-14 \end{bmatrix}$$

∴ which is in normal form

$\rho(A) = 3$ if $14k-14 = 0$ i.e $k=1$

$\rho(A) = 4$ if $14k-14 \neq 0$ i.e $k \neq 1$

→ Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ to normal form and hence find the rank.

Sol. Let $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Reduce the matrix A into normal form by applying row and column operations.

$C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$C_3 \to C_3 - 2C_1 \quad C_4 \to C_4 + 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 2C_3 - C_2, \quad C_4 \to 2C_4 - 3C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2\left(\frac{1}{4}\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Which is in normal form

$\therefore \rho(A) = 2$.

→ By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

sol: Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

Reduce the matrix A into normal form by applying elementary row and column operations.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -12 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$ $C_3 \to C_3 - 3C_1$ $C_4 \to C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$C_3 \to 3C_3 - 2C_2$, $C_4 \to 3C_4 - 5C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -36 \end{bmatrix}$$

$R_2 \to R_2\left(-\frac{1}{3}\right)$ $R_3 \to R_3\left(-\frac{1}{36}\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

Which is in normal form

$\therefore \quad \rho(A) = 3.$

Elementary Matrix :-

It is a matrix obtained from a unit matrix by a single elementary transformation.

Eg:- $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the elementary matrices obtained from I_3 by applying the elementary operations $E_1 \leftrightarrow C_3$, $R_1 \to R_1(3)$ and $R_1 \to R_1 + 3R_2$, respectively.

Theorem :-

Every elementary row (column) transformation of a matrix can be obtained by pre multiplication (post-multiplication) with corresponding elementary matrix.

Eg :- Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 9 \\ -2 & 5 & 6 \end{bmatrix}$

Let us interchange 1^{st} and 3^{rd} rows, we get $B = \begin{bmatrix} -2 & 5 & 6 \\ 2 & 3 & 9 \\ 1 & 3 & 5 \end{bmatrix}$

This B is same as the matrix obtained by pre multiplying A with the matrix E_{13} obtained from unit matrix by interchanging 1^{st} and 3^{rd} rows in it.

Verification :- $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$E_{13} A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 9 \\ -2 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 6 \\ 2 & 3 & 9 \\ 1 & 3 & 5 \end{bmatrix}$$

Eg :- $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 9 \\ -2 & 5 & 6 \end{bmatrix}$

Let us Interchange 1^{st} and 3^{rd} columns, we get $B = \begin{bmatrix} 5 & 3 & 1 \\ 9 & 3 & 2 \\ 6 & 5 & -2 \end{bmatrix}$

This B is same as the matrix obtained by **post** multiplying A with the matrix E'_{13} obtained from unit matrix by Interchanging 1^{st} and 3^{rd} columns in it.

<u>Verification</u> :- $E'_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$AE'_{13} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 9 \\ -2 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 1 \\ 9 & 3 & 2 \\ 6 & 5 & -2 \end{bmatrix}$$

PAQ form of a Matrix :-

If A be an $m\times n$ matrix of rank-r, then there exists two non singular matrices P and Q such that $PAQ=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called PAQ form of a matrix A.

Working procedure :-

Consider the matrix $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

We can write $A_{3\times3} = I_3 A I_3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we have to reduce the matrix A on the L.H.S to the normal form by applying elementary transformations.

Each row transformation will be applied to the pre factor I_3 and each column transformation will be applied to the post factor I_3 on the R.H.S of equation ①.

Step (i) :- If $a_{11}\neq 0$, by using a_{11} position make a_{21} and a_{31} positions as zero. Here we apply row operations. The same row operations apply prefactor of A on R.H.S of ①.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step (ii) :- By using a_{11} position make a_{12} and a_{13} positions as zero. Here we apply column operations. The same column operations apply post factor of A.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & a''_{23} \\ 0 & a''_{32} & a''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

step(iii) :- If $a''_{22} \neq 0$, by using a''_{22} position make a''_{32} position as zero. Here we apply row operation. The same row operation apply on prefactor of A.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & a''_{23} \\ 0 & 0 & a'''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

step(iv) :- By using a''_{22} position make a''_{23} position as zero. Here we apply column operation. The same column operation apply on post factor of A.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & 0 \\ 0 & 0 & a^{iv}_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Step (v) :- By using elementary transformations reduce the matrix on L.H.S to an identity matrix. The same operations apply on prefactor or post factor on R.H.S.

The resultant is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

Where P and Q non singular matrices.

Note :- Here the non singular matrices P and Q are not unique.

→ Obtain the non singular matrices P and Q such that PAQ is in the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ Also find the rank of the matrix A.

Sol: Given that $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}_{3\times3}$.

We can write $A = I_3 A I_3$ ——— ①

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we have to reduce the matrix A on the L.H.S to the normal form by applying elementary transformations.

Each elementary row transformation will be applied to the prefactor I_3 and each elementary column transformation will be applied to the post factor I_3 of the R.H.S of equation ①.

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1 \quad C_3 \to C_3 - C_1$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \longrightarrow C_2 - C_1 \quad C_3 \longrightarrow C_3 - C_1 \quad C_4 \longrightarrow C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \longrightarrow 5R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & -2 \\ 0 & 0 & -24 & -48 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \longrightarrow 5C_3 - C_2 \quad , \quad C_4 \longrightarrow 5C_4 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -120 & -240 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & -8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$C_4 \longrightarrow C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -120 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 5 & -1 & -8 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$R_2 \longrightarrow R_2\left(-\frac{1}{5}\right) , \quad R_3 \longrightarrow R_3\left(-\frac{1}{120}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/5 & 2/5 \\ -\frac{1}{24} & \frac{1}{120} & \frac{1}{15} \end{bmatrix} A \begin{bmatrix} 1 & -1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

This is of the form $\begin{bmatrix} I_3 & 0 \end{bmatrix} = PAQ$

Where $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/5 & 2/5 \\ -\frac{1}{24} & \frac{1}{120} & \frac{1}{15} \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -1 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Here P and Q are non singular matrices.

$\therefore \rho(A) = 3$.

PAQ Form of a Matrix

1 Find the matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$ Ans: 2.

2 Find the matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$ Ans:- 2.

3 Find the non singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$$ Ans:- 2

4 Find the non singular matrices P and Q such that PAQ is in the normal form Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$ Ans:- 3.

5 Find the non singular matrices P and Q such that PAQ is in the normal form Hence find the rank of A.

$$A = \begin{bmatrix} 4 & -3 & 1 \\ 1 & -1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$ Ans:- 3

6 Find the non singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$$ Ans:- 3.

The Inverse of a Non Singular Matrix by Elementary Transformations :-

(Gauss Jordan Method) :-

We can find the inverse of a non singular matrix by using elementary row operations only. This method is known as Gauss Jordan Method.

If a non singular matrix A of order n is reduced to the unit matrix I_n by sequence of E-row transformations only, then the same sequence of E-row transformations applied to the unit matrix I_n gives the inverse of A i.e A^{-1}.

Working Procedure to find inverse of non singular matrix by using row operations :-

Suppose A is a non singular matrix of order 3.

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{We write} \quad A_{3\times3} = I_3 A \quad \text{—— (1)}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Now we reduce the matrix A on the L.H.S to be identity matrix I_3 by applying E-row transformations only. Each E-row transformation will be applied to the prefactor I_3 of the R.H.S of eqn (1).

Step(i) :- If $a_{11} \neq 0$, by using a_{11} position make a_{21} and a_{31} positions as zero. Here we apply row operations. The same operations apply on prefactor of A on R.H.S of (1)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A$$

Step (ii) :- If $a'_{22} \neq 0$, by using a'_{22} position make a_{12} and a'_{32} positions as zero. Here we apply row operations. The same operations apply on prefactor of A.

$$\begin{bmatrix} a'_{11} & 0 & a'_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A$$

Step (iii) :- If $a''_{33} \neq 0$, by using a''_{33} position make a'_{23} and a'_{13} positions as zero. Here we apply row operations. The same operations apply on prefactor of A.

$$\begin{bmatrix} a''_{11} & 0 & a \\ 0 & a''_{22} & 0 \\ 0 & 0 & a''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A$$

Step (iv) :- $R_1 \to R_1\left(\frac{1}{a''_{11}}\right)$, $R_2 \to R_2\left(\frac{1}{a''_{22}}\right)$, $R_3 \to R_3\left(\frac{1}{a''_{33}}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = BA$$

$$I = BA$$

$\therefore$ B is called inverse of A.

→ Find the inverse of the matrix $A=\begin{bmatrix}2 & -1 & 3\\ 1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}$ by using elementary transformations.. verify $AA^{-1}=I$

Sol:- Let $A=\begin{bmatrix}2 & -1 & 3\\ 1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}$

We can write $A = I_3 . A$ ——— ①

$$\begin{bmatrix}2 & -1 & 3\\ 1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

Now reduce the matrix A on L.H.S to the Identity matrix I_3 by using E-row transformations only. Each row transformation will be applied to the prefactor I_3 of the R.H.S of equation ① .

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & 3\\ 1 & -1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

$R_2 \rightarrow R_2-2R_1$ $R_3 \rightarrow R_3-R_1$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & -3 & 1\\ 0 & -2 & 0\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 1 & -2 & 0\\ 0 & -1 & 1\end{bmatrix}A$$

$R_1 \rightarrow 3R_1+R_2$ $R_3 \rightarrow 3R_3-2R_2$

$$\begin{bmatrix}3 & 0 & 4\\ 0 & -3 & 1\\ 0 & 0 & -2\end{bmatrix}=\begin{bmatrix}1 & 1 & 0\\ 1 & -2 & 0\\ -2 & 1 & 3\end{bmatrix}A$$

$R_1 \rightarrow R_1+2R_3$ $R_2 \rightarrow 2R_2+R_3$

$$\begin{bmatrix}3 & 0 & 0\\ 0 & -6 & 0\\ 0 & 0 & -2\end{bmatrix}=\begin{bmatrix}-3 & 3 & 6\\ 0 & -3 & 3\\ -2 & 1 & 3\end{bmatrix}A$$

$R_1 \to R_1\left(\frac{1}{3}\right) \quad R_2 \to R_2\left(\frac{-1}{6}\right) \quad R_3 \to R_3\left(\frac{-1}{2}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A$$

Which is of the form $I_3 = BA$

Here B is called Inverse of A. [$\because$ By def.]

$$\therefore \quad B = A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

Verification :-

$$AA^{-1} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore AA^{-1} = I$.

Working Procedure to find Inverse of non singular matrix by using column operations :-

Suppose A is a non singular matrix of order 3.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

We write $A_{3\times3} = A I_3$ —— ①

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we reduce the matrix A on the L.H.S to be identity matrix I_3 by applying E-column transformations only. Each E-column transformation will be applied to the post factor I_3 of the R.H.S of eqn ①.

Step (i) :- If $a_{11} \neq 0$, by using a_{11} position make a_{12} and a_{13} positions as zero. Here we apply column operations. The same operations apply on post factor of A on R.H.S of ①.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a'_{22} & a'_{23} \\ a_{31} & a'_{32} & a'_{33} \end{bmatrix} = A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Step (ii) :- If $a'_{22} \neq 0$ by using a'_{22} position make a_{21} and a'_{23} positions as zero. Here we apply column operations. The same operations apply on post factor of A

$$\begin{bmatrix} a'_{11} & 0 & 0 \\ 0 & a'_{22} & 0 \\ a'_{31} & a'_{32} & a''_{33} \end{bmatrix} = A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

step(iii) :- If $a''_{33} \neq 0$, by using a''_{33} position make a'_{31}, a'_{32} positions as zero. Here we apply column operations. The same operations apply on ~~post~~factor of A

$$\begin{bmatrix} a''_{11} & 0 & 0 \\ 0 & a''_{22} & 0 \\ 0 & 0 & a''_{33} \end{bmatrix} = A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

step(iv) :- $C_1 \rightarrow C_1\left(\frac{1}{a''_{11}}\right)$ $C_2 \rightarrow C_2\left(\frac{1}{a''_{22}}\right)$ $C_3 \rightarrow C_3\left(\frac{1}{a'_{33}}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AB$$

$$I = BA$$

$\therefore$ B is called inverse of A.

Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ by using elementary column transformations. Verify $AA^{-1} = I$.

Sol:- Given that $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

We can write $A = AI_3$

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we reduce the matrix A on L.H.S to the Identity matrix I_3 by using E-column transformations only. Each column transformation will be applied to the post factor I_3 of the R.H.S of equation ①.

$C_2 \to 2C_2 + C_1 \quad C_3 \to 2C_3 - 3C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$C_1 \to 3C_1 - C_2 \; ; \; C_3 \to 3C_3 + C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 4 & -1 & -4 \end{bmatrix} = A \begin{bmatrix} 2 & 1 & -8 \\ -2 & 2 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

$C_1 \to C_1 + C_3 \quad C_2 \to 4C_2 - C_3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -4 \end{bmatrix} = A \begin{bmatrix} -6 & 12 & -8 \\ 0 & 6 & 2 \\ 6 & -6 & 6 \end{bmatrix}$$

$C_1 \to C_1\left(\frac{1}{6}\right)$, $C_2 \to C_2\left(\frac{1}{12}\right)$, $C_3 \to C_3\left(-\frac{1}{4}\right)$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

which is of the form $I_3 = AB$.

Here B is called inverse of A. [$\because$ By def]

$$\therefore \quad B = A^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

Verification :-

$$AA^{-1} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad AA^{-1} = I.$$

INVERSE OF MATRIX

1 Define Inverse of matrix

2 Employing elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ Ans: $A^{-1} = \frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$

3 Employing elementary column transformations, find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$ Ans: $A^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

4 Find the inverse of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ by elementary column transformations. Ans: $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

5 Employing elementary row transformations find the inverse of the matrix $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ Ans: $A^{-1} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 6 \end{bmatrix}$

6 Find the inverse of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ by using elementary row transformations.

Ans:- $A^{-1} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$

7 Find the inverse of matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ by using elementary column transformations.

Ans:- $\begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$

System of simultaneous linear non-homogeneous equations :-

A system of m simultaneous non homogeneous linear equations in n unknowns $x_1, x_2, x_3, \ldots x_n$ is of the form.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{———} (1)$$

We can write the above system of equations (1) in the form of matrix equation given by $AX = B$ ——— (2)

i.e $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$ is called the coefficient matrix of the system of equations (1)

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ is the matrix of unknowns and .

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$ be the constant matrix of the system of equations (1)

The set of values $x_1, x_2, x_3, \ldots x_n$ which satisfy the system (1) is called the solution of the system.

Augmented Matrix :—

The matrix $[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & ; & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & : & : \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & ; & b_m \end{bmatrix}$ is said to be the augmented matrix of the given system of non homogeneous equations.

Consistency and In consistency :—

Any system of equations which contains one or more solutions is said to be consistent otherwise it is said to be inconsistent i.e the inconsistent system does not contain any solution.

Condition for consistency (Rank Test) :—

The necessary and sufficient condition for a system of non homogeneous equations $AX = B$ is said to be consistent is that the rank of the co efficient matrix A is same as the rank of the augmented matrix $[A|B]$. Then the system of equations $AX = B$ is consistent $\Longleftrightarrow \rho(A) = \rho([A|B])$

Note :- If $\rho(A) \neq \rho([A|B])$ then the given system $AX = B$ is inconsistent

Working Procedure :—

Suppose we have m equations in n unknowns.

The matrix equation of the given system of equations is $AX = B$. Then the co efficient matrix A is of order $m \times n$. Now write the augmented matrix $[A|B]$.

Step 1 :- First reduce the augmented matrix $[A|B]$ to echolon form by applying E-row operations only. With this we get the ranks of the augmented matrix $[A|B]$ and the co efficient matrix A.

Step 2 :-

Case (i) :- When $\rho(A) \neq \rho([A|B])$

In this case the given system of equations i.e $AX = B$ is inconsistent i.e it has no solution.

Case (ii) :- When $\rho(A) = \rho([A|B]) = r$ say

In this case the given system of equations i.e $AX = B$ is consistent i.e it contains a solution.

Now we have to verify the following points.

(a) If $r = n$ i.e the no. of unknowns then the given system has a unique solution.

(b) If $r < n$ i.e the no. of unknowns, the given system contains an infinite no. of solutions. To determine these solutions we have to assign an arbitrary value for $(n-r)$ variables and the remaining are depending upon them.

(c). If $m < n$ i.e the no. of equations less than the no. of unknowns then since $r \leq m < n$, the given system possesses an intinitinite no. of solutions.

Properties :-

Symmetric and skew symmetric matrices :—

Theorem: A necessary and sufficient condition for a matrix A to be symmetric is that $A^T = A$. (OR) A is symmetric $\Leftrightarrow A^T = A$.

Proof :- A is symmetric $\Rightarrow A^T = A$.

Let $A = [a_{ij}]$ be an n- row square matrix so that $a_{ij} = a_{ji}$

Also A^T is also an n- rowed square matrix and

the $(i,j)^{th}$ element of A^T = the $(j,i)^{th}$ element of A.

$= a_{ji}$

$= a_{ij} = (i,j)^{th}$ element of A.

Hence $A^T = A$.

Converse: $A^T = A \Rightarrow$ A is symmetric.

Now $(i,j)^{th}$ element of A = $(i,j)^{th}$ element of A^T (Given $A^T = A$)

$= (j,i)^{th}$ element of A.

Hence A is symmetric matrix.

Theorem: A necessary and sufficient condition for a matrix A to be skew symmetric matrix is that $A^T = -A$ (OR) A is skew symmetric $\Leftrightarrow A^T = -A$

Proof :- A is skew symmetric $\Rightarrow A^T = -A$.

Let A be an n- rowed skew symmetric matrix.

So that $a_{ij} = -a_{ji}$

Now A^T is also n- rowed square matrix $(i,j)^{th}$ element of A^T

$= (j,i)^{th}$ element of A.

$a_{ji} = -a_{ij}$ = the $(i,j)^{th}$ element of $-A$.

Hence $A^T = -A$.

<u>Converse</u> : $A^T = -A \implies A$ is skew symmetric

Now $(i,j)^{th}$ element of A = the negative of $(i,j)^{th}$ element of A^T

= the negative of $(j,i)^{th}$ element of A

A is a skew symmetric matrix.

<u>Theorem</u> :- The inverse of a non singular matrix A is symmetric.

<u>Proof</u> :- A is non singular matrix $\implies A^{-1}$ exists $\implies A^T = A$.

Now $(A^{-1})^T = (A^T)^{-1} = A^{-1}$

$(A^{-1})^T = A^{-1} \implies A^{-1}$ is symmetric.

<u>Theorem</u> :- If A and B are symmetric matrices then AB is symmetric if and only if $AB = BA$.

<u>Proof</u> :- Given A and B are symmetric $\implies A^T = A$ and $B^T = B$.

Suppose $AB = BA$

Consider $(AB)^T = B^T A^T = BA = AB$.

$(AB)^T = AB \implies AB$ is symmetric.

<u>Conversely</u>, suppose AB is symmetric.

$\implies AB = (AB)^T = B^T A^T = BA$.

$\implies AB = BA$.

Hence AB is symmetric if and only if $AB = BA$.

<u>Theorem</u> :- If A be any matrix then AA^T and A^TA are both symmetric matrices

<u>Proof</u> :- Let A be any matrix.

Now $(AA^T)^T = (A^T)^T A^T = AA^T \implies AA^T$ is symmetric.

Also $(A^T A)^T = A^T (A^T)^T = A^T A \implies A^T A$ is symmetric.

Theorem:- The matrix $B^T A B$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.

Proof:- (i) Let A be symmetric matrix $\Rightarrow A^T = A$.

Now $(B^T A B)^T = B^T A^T (B^T)^T = B^T A B$

$\Rightarrow B^T A B$ is symmetric.

(ii) Let A be skew symmetric matrix $\Rightarrow A^T = -A$.

Now $(B^T A B)^T = B^T A^T (B^T)^T = B^T(-A)B = -B^T A B$.

$\therefore \quad (B^T A B)^T = -B^T A B$

$\Rightarrow \quad B^T A B$ is skew symmetric.

Theorem:- All positive integral powers of a symmetric matrix are symmetric.

Proof:- Let A be symmetric matrix.

Now $A^n = A.A.A \ldots A$ upto n times where n is a +ve integer.

$(A^n)^T = (A.A.A \ldots A \text{ upto } n \text{ times})^T$

$= A^T A^T A^T \ldots A^T$ upto n times.

$= A\, A\, A \ldots A$ upto n times.

$= A^n$

$\therefore \quad (A^n)^T = A^n \Rightarrow A^n$ is symmetric.

Theorem:- Positive odd integral powers of a skew symmetric matrix are skew symmetric where as positive even integral powers are symmetric.

Proof: Let A be a skew symmetric matrix $\Rightarrow A^T = -A$

Now $(A^n)^T = (A.A.A \ldots A \text{ n times})^T = A^T A^T A^T \ldots A^T$ n times

$= (-A)(-A)(-A) \ldots (-A)$ n times.

$= (-1)^n A^n$ where n is a +ve integer

$= -A^n$ or A^n according as n is odd or even.

If n is an odd +ve integer, then $(A^n)^T = -A^n \Rightarrow A^n$ is skew symmetric.

If n is an even +ve integer then $(A^n)^T = A^n \Rightarrow A^n$ is Symmetric.

Properties of orthogonal matrix :—

Theorem :- If A is orthogonal matrix, then $|A| = \pm 1$

Proof :- Given A is orthogonal matrix $\Rightarrow A^T A = I$.

$$\Rightarrow |A^T A| = |I|.$$

$$\Rightarrow |A^T||A| = 1.$$

$$|A||A| = 1 \qquad [\because |A^T| = |A|].$$

$$|A|^2 = 1$$

$$|A| = \pm 1.$$

Since $|A| \neq 0$, A is invertible.

Now $A^T A = I \Rightarrow A^T (A A^{-1}) = I A^{-1}$

$$A^T I = A^{-1}$$

$$A^T = A^{-1}.$$

Note :- A is orthogonal $\Rightarrow A A^T = I = A^T A$.

A is orthogonal $\Rightarrow A^T = A^{-1}$

Theorem : If A, B be orthogonal matrices. AB and BA are also orthogonal.

Proof :- Let A and B are n-rowed square matrices.

$|AB| = |A|\,|B| \Rightarrow |AB| \neq 0$ since $|A| \neq 0$ and $|B| \neq 0$.

$$(AB)^T = B^T A^T$$

$$(AB)^T (AB) = (B^T A^T)(AB)$$

$$= B^T (A^T A) B$$

$$= B^T I B$$

$$= B^T B$$

$$= I.$$

$[\because$ A is orthogonal

$AA^T = A^T A = I$

B is orthogonal

$BB^T = B^T B = I$

$$(AB)^T (AB) = I$$

$\Rightarrow AB$ is orthogonal.

Similarly $(AB)(AB)^T = I$

$\Rightarrow AB$ is orthogonal.

$$(BA)(BA)^T = (BA)(A^T B^T)$$
$$= B(AA^T)B^T$$
$$= BIB^T$$
$$= BB^T$$
$$(BA)(BA)^T = I$$
$\Rightarrow BA$ is orthogonal

Similarly $(BA)^T(BA) = (A^T B^T)(BA)$
$$= A^T(B^T B)A$$
$$= A^T I A$$
$$= A^T A$$
$$(BA)^T(BA) = I$$
$\Rightarrow BA$ is orthogonal.

Verify that the determinant of an orthogonal matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is ± 1

Sol: Given that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$|A| = \cos^2\theta + \sin^2\theta = 1$$

$$|A| = 1.$$

If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ are orthogonal matrices then prove that AB and BA are orthogonal.

Sol:- Given that $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

$$AB = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$AB = \frac{1}{3}\begin{bmatrix} \cos\theta - 2\sin\theta & 2 & \sin\theta + 2\cos\theta \\ 2\cos\theta + 2\sin\theta & 1 & 2\sin\theta - 2\cos\theta \\ -2\cos\theta + \sin\theta & 2 & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(AB)^T = \frac{1}{3}\begin{bmatrix} \cos\theta - 2\sin\theta & 2\cos\theta + 2\sin\theta & -2\cos\theta + \sin\theta \\ 2 & 1 & 2 \\ \sin\theta + 2\cos\theta & 2\sin\theta - 2\cos\theta & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(AB)(AB)^T = \frac{1}{9}\begin{bmatrix} \cos\theta - 2\sin\theta & 2 & \sin\theta + 2\cos\theta \\ 2\cos\theta + 2\sin\theta & 1 & 2\sin\theta - 2\cos\theta \\ -2\cos\theta + \sin\theta & 1 & -2\sin\theta - \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta - 2\sin\theta & 2\cos\theta + 2\sin\theta & -2\cos\theta + \sin\theta \\ 2 & 1 & 2 \\ \sin\theta + 2\cos\theta & 2\sin\theta - 2\cos\theta & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(AB)(AB)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

$\therefore$ AB is an orthogonal matrix.

$$BA = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$BA = \frac{1}{3}\begin{bmatrix} \cos\theta - 2\sin\theta & 2\cos\theta + 2\sin\theta & 2\cos\theta - \sin\theta \\ 2 & 1 & -2 \\ -\sin\theta - 2\cos\theta & -2\sin\theta + 2\cos\theta & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(BA)^T = \frac{1}{3}\begin{bmatrix} \cos\theta - 2\sin\theta & 2 & -\sin\theta - 2\cos\theta \\ 2\cos\theta + 2\sin\theta & 1 & -2\sin\theta + 2\cos\theta \\ 2\cos\theta - \sin\theta & -2 & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(BA)(BA)^T = \frac{1}{9}\begin{bmatrix} \cos\theta - 2\sin\theta & 2\cos\theta + 2\sin\theta & 2\cos\theta - \sin\theta \\ 2 & 1 & -2 \\ -\sin\theta - 2\cos\theta & -2\sin\theta + \cos\theta & -2\sin\theta - \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta - 2\sin\theta & 2 & -\sin\theta - 2\cos\theta \\ 2\cos\theta + 2\sin\theta & 1 & -2\sin\theta + \cos\theta \\ 2\cos\theta - \sin\theta & -2 & -2\sin\theta - \cos\theta \end{bmatrix}$$

$$(BA)(BA)^T = I.$$

BA is an orthogonal matrix.

Theorem :- The inverse of an orthogonal matrix is orthogonal.

proof :- Let A be an orthogonal matrix $\Rightarrow AA^T = I = A^TA$

Taking inverse $\Rightarrow (AA^T)^{-1} = I^{-1} = (A^TA)^{-1}$

$$(A^T)^{-1}A^{-1} = I = A^{-1}(A^T)^{-1}$$

$$(A^{-1})^T A^{-1} = I = (A^{-1})(A^{-1})^T$$

$\Rightarrow A^{-1}$ is an orthogonal

Theorem :- The transpose of an orthogonal matrix is orthogonal.

proof :- Let A be an orthogonal matrix $\Rightarrow AA^T = I = A^TA$.

Taking transpose $\Rightarrow (AA^T)^T = I^T = (A^TA)^T$

$$(A^T)^T A^T = I = A^T (A^T)^T$$

$\Rightarrow A^T$ is orthogonal.

Eg:- Proove that inverse of an orthogonal matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogo-nal.

Sol Given that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$|A| = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

$$A^{-1} = \frac{1}{|A|} \text{Adj} A$$

$$A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$(A^{-1})(A^{-1})^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$(A^{-1})(A^{-1})^T = I.$$

$\therefore A^{-1}$ is an orthogonal.

Eg:- Proove that transpose of an orthognal matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

sol:- Given that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$(A^T)(A^T)^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$(A^T)(A^T)^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ A^T is an orthogonal.

Idempotent Matrix :—

A square matrix A is said to be Idempotent if $A^2 = A$.

Eg:- $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$

$$A^2 = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix} = A.$$

$$A^2 = A$$

$\therefore$ A is an idempotent matrix.

1) show that the matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Nilpotent Matrix :—

If A is a square matrix such that $A^m = 0$ where m is a least positive integer then A is called nilpotent.

If m is least +ve integer such that $A^m = 0$ then A is called nilpotent of index m.

Eg:- $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 = 0$.

$\therefore$ A is a nilpotent matrix of index 3.

(1) s/t for any real values of a and b the matrix $\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent of index 2.

(2) s/T for $a \neq 0, b \neq 0$ the matrix $\begin{bmatrix} a & -b & -(a+b) \\ -a & b & a+b \\ a & -b & -(a+b) \end{bmatrix}$ is a nilpotent matrix index 2.

Involutary Matrix :-

If A is a square matrix such that $A^2 = I$ (I is unit matrix of order same as that of A) then A is said to be Involutary.

Eg :- $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = I$$

$\Rightarrow$ A is involutary.

(1) S/T $A = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}$ is involutary.

(2) S/T $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ is involutary.

Periodic Matrices :—

If A is a square matrix such that $A^{n+1} = A$ where n is a +ve integer then A is called a periodic matrix.

If n is the least +ve integer satisfying the relation $A^{n+1} = A$ then n is called the period of A.

Eg:- $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$A^2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^2 = A.$$

$\therefore$ A is periodic of order one.

(1) PT $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a periodic matrix and its period is 4.

DETERMINANTS :—

Determinant of a 2×2 matrix :—

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix of order 2, then the value $ad-bc$ is called the determinant of A. It is denoted by det A or $|A|$

i.e $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$.

Eg:- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $|A| = 4-6 = -2$

Minors and Cofactors of a Square matrix :—

(a) The minor of an element a_{ij} in a determinant is obtained by omitting the row and the column of the a_{ij}. It is denoted by M_{ij}.

(b) The cofactor of an element a_{ij} in a determinant is obtained by multiplying its minor with $(-1)^{i+j}$. where i, j indicate the row and column of the element a_{ij}. It is denoted by A_{ij}.

i.e $A_{ij} = (-1)^{i+j} M_{ij} \quad \forall i,j$

Eg :- If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 7 \\ -6 & 5 & 8 \end{bmatrix}$

(i) The minor of an element 4 is

$$M_{22} = \begin{vmatrix} 1 & 3 \\ -6 & 8 \end{vmatrix} = 8+18 = 26.$$

(ii) The cofactor of an element 4 is

$$A_{22} = (-1)^{2+2} M_{22} = 26.$$

Determinant of an $n \times n$ matrix :—

The sum of the products of the elements of any row or any column by its corresponding cofactors is said to be the determinant of a matrix of order n.

We can expand the determinant in terms of any row or any column of the matrix.

Thus if $A = [a_{ij}]_{n \times n}$ then

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \cdots + a_{1n}A_{1n}$$

$$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + \cdots + a_{2n}A_{2n}$$

$$\cdots$$

$$= a_{n1}A_{n1} + a_{n2}A_{n2} + a_{n3}A_{n3} + \cdots + a_{nn}A_{nn}.$$

$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \cdots + a_{n1}A_{n1}.$$

$$\cdots$$

$$= a_{1n}A_{1n} + a_{2n}A_{2n} + a_{3n}A_{3n} + \cdots + a_{nn}A_{nn}.$$

Thus if $A = [a_{ij}]_{3 \times 3}$ then

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

$$= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$

$$\cdots$$

$$|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

Where $A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$\therefore |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Find the determinant of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$|A| = 8 \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} - (-6) \begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14)$$

$$|A| = 40 - 60 + 20$$

$$|A| = 0.$$

Note :— (i) If A is a square matrix of order n and k is any scalar then $|kA| = k^n |A|$.

(ii) If A is a square matrix of order n, Then $|A| = |A^T|$.

(iii) If A and B be two square matrices of same order Then $|AB| = |A||B|$.

Eg : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ Then

$$|A| = a_{11}.A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 1(-2-4) + 2(4-12) + 3(2+3)$$

$$|A| = -6 + 16 + 15 = 25.$$

Adjoint of a Matrix :—

If A is a square matrix of order n, then the transpose of the cofactor matrix of A is said to be the adjoint of a matrix A. It is denoted by adj A.

Thus if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the cofactor matrix of

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\therefore \text{Adj } A = \left[\text{The cofactor matrix of } A\right]^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Note :- If A is a square matrix of order n, then

$A(\text{adj}A) = (\text{adj}A).A = |A|.I$ where I is a unit matrix of order n.

Eg:- $A = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -3 & -1 \\ -4 & 1 & 3 \end{bmatrix}$

Cofactor of an element $a_{23} = -1$ is $A_{23} = (-1)^{2+3} \begin{vmatrix} 6 & 2 \\ -4 & 1 \end{vmatrix}$

$A_{23} = -(6+8)$

$A_{23} = -14$

Cofactor of an element $a_{31} = -4$ is $A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix}$

$A_{31} = -2+12$

$A_{31} = 10$

Cofactor of an element $a_{22} = -3$ is $A_{22} = (-1)^{2+2} \begin{vmatrix} 6 & 4 \\ -4 & 3 \end{vmatrix}$

$A_{22} = 18+16$

$A_{22} = 34$

Cofactor of an element $a_{12} = 2$ is $A_{12} = (-1)^{1+2} \begin{vmatrix} -2 & -1 \\ -4 & 3 \end{vmatrix}$

$A_{12} = -(-6-4)$

$A_{12} = 10$

<u>Inverse of a Matrix</u> :-

Let A be any square matrix then a matrix B if exists such that $AB = BA = I$ then B is called inverse of A and is denoted by A^{-1}.

<u>Singular matrix</u> :- A square matrix A is said to be singular if $|A| = 0$.

<u>Non singular matrix</u> :- A square matrix A is said to be non singular if $|A| \neq 0$.

→ Thus only non singular matrices possess inverses.

Theorem :- The necessary and sufficient condition for a square matrix to possess inverse is that $|A| \neq 0$.

Note :- If $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

Find the inverse of $A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$

We have $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$$|A| = \begin{vmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{vmatrix}$$

$$= 7(-6+4) - 2(0-3) + 1(0+9)$$

$$= -14 + 6 + 9$$

$$|A| = 1$$

Cofactor of an element $a_{11} = 7$ is $A_{11} = \begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}(-1)^{1+1} = -2$

Cofactor of an element $a_{12} = 2$ is $A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & -1 \\ -3 & -2 \end{vmatrix} = 3$

Cofactor of an element $a_{13} = 1$ is $A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 3 \\ -3 & 4 \end{vmatrix} = 9$

Cofactor of an element $a_{21} = 0$ is $A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = 8$

Cofactor of an element $a_{22} = 3$ is $A_{22} = (-1)^{2+2}\begin{vmatrix} 7 & 1 \\ -3 & -2 \end{vmatrix} = -11$

Cofactor of an element $a_{23} = -1$ is $A_{23} = (-1)^{2+3}\begin{vmatrix} 7 & 2 \\ -3 & 4 \end{vmatrix} = -34$

Cofactor of an element $a_{31} = -3$ is $A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -5$

Cofactor of an element $a_{32} = 4$ is $A_{32} = (-1)^{3+2}\begin{vmatrix} 7 & 1 \\ 0 & -1 \end{vmatrix} = 7$

cofactor of an element $a_{33} = -2$ is $A_{33} = (-1)^{3+3}\begin{vmatrix} 7 & 2 \\ 0 & 3 \end{vmatrix} = 21$

$$\text{Cofactor matrix of } A = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}$$

$$\text{adj } A = \left[\text{Cofactor matrix of } A\right]^T = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

We have $A^{-1} = \dfrac{1}{|A|} \text{adj } A$

$$\therefore \quad A^{-1} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

Matrix inversion Method :-

The system of linear equations are

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad \text{--- (1)}$$

The matrix form of given system of equations is $AX = B$.

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The solution of the given system is $X = A^{-1}B$.

Solve $7x + 2y + z = 21$, $3y - z = 5$, $-3x + 4y - 2z = -1$, by Matrix inversion method.

sol: Given that $7x + 2y + z = 21$

$$3y - z = 5$$

$$-3x + 4y - 2z = -1$$

The matrix form of given system of equations is $AX = B$.

where $A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix}$

The solution of system of equations by matrix inversion method is $X = A^{-1}B$

where $A^{-1} = \frac{1}{|A|} \operatorname{adj} A$.

$$|A| = \begin{vmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{vmatrix} = 7(-6+4) - 2(0-3) + 1(0+9) = 1$$

Cofactor matrix of $A = \begin{bmatrix} -2 & 3 & 9 \\ 8 & -11 & -34 \\ -5 & 7 & 21 \end{bmatrix}$

$$\text{adj}\,A = \left[\text{Cofactor matrix of } A\right]^T = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

We have $A^{-1} = \dfrac{1}{|A|}\,\text{adj}\,A$

$$\therefore\ A^{-1} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}\begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} -42+40+5 \\ 63-55-7 \\ 189-170-21 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Which is the solution of the given system of eqn's.

CRAMER'S RULE (DETERMINANT METHOD):-

The given system of linear equations are

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{——} (1)$$

The matrix form of the system (1) is $AX = B$.

Where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The solution of the system (1) is given by

$$x = \frac{\Delta_1}{\Delta} \quad y = \frac{\Delta_2}{\Delta} \quad z = \frac{\Delta_3}{\Delta} \quad (\Delta \neq 0)$$

Where $\Delta = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

We notice that $\Delta_1, \Delta_2, \Delta_3$ are the determinants obtained from Δ on replacing the 1st, 2nd and 3rd columns by d's i.e (d_1, d_2, d_3) respectively.

Solve $-x+3y-2z=5$, $4x-y-3z=-8$, $2x+2y-5z=7$ by Cramer's rule.

Sol:- Given that $-x+3y-2z=5$, $4x-y-3z=-8$ $2x+2y-5z=7$.

The matrix form of given System of eqn's is $AX=B$

Where $A = \begin{bmatrix} -1 & 3 & -2 \\ 4 & -1 & -3 \\ 2 & 2 & -5 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ -8 \\ 7 \end{bmatrix}$

The solution of linear system of equations by cramer's rule is given by $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$ $z = \frac{\Delta_3}{\Delta}$.

$$\Delta = |A| = \begin{vmatrix} -1 & 3 & -2 \\ 4 & -1 & -3 \\ 2 & 2 & -5 \end{vmatrix} = -1(5+6) - 3(-20+6) - 2(8+2)$$

$$\Delta = |A| = -11 + 42 - 20 = 11$$

$$\Delta_1 = \begin{vmatrix} 5 & 3 & -2 \\ -8 & -1 & -3 \\ 7 & 2 & -5 \end{vmatrix} = 5(5+6) - 3(40+21) - 2(-16+7)$$

$$\Delta_1 = 55 - 183 + 18 = -110$$

$$\Delta_2 = \begin{vmatrix} -1 & 5 & -2 \\ 4 & -8 & -3 \\ 2 & 7 & -5 \end{vmatrix} = -1(40+21) - 5(-20+16) - 2(28+16)$$

$$\Delta_2 = -61 + 70 - 88 = -79$$

$$\Delta_3 = \begin{vmatrix} -1 & 3 & 5 \\ 4 & -1 & -8 \\ 2 & 2 & 7 \end{vmatrix} = -1(-7+16) - 3(28+16) + 5(8+2)$$

$$\Delta_3 = -9 - 132 + 50$$

$$\Delta_3 = 91$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-110}{11} = -10 \qquad y = \frac{\Delta_2}{\Delta} = \frac{-79}{11} \qquad z = \frac{\Delta_3}{\Delta} = \frac{-91}{11}$$

$\therefore$ The solution of the given system of equations is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ -79/11 \\ -91/11 \end{bmatrix}$$

Sub Matrix :— A matrix obtained by deleting a row or a column or both of a given matrix is called its sub matrix of the given matrix.

Eg :- Let $A = \begin{bmatrix} 1 & 3 & -4 & 7 & 8 \\ 9 & 8 & 2 & 8 & 7 \\ 5 & 6 & 9 & 5 & 3 \end{bmatrix}_{3\times5}$

Then $\begin{bmatrix} 1 & 3 & 7 & 8 \\ 9 & 8 & 8 & 7 \\ 5 & 6 & 5 & 3 \end{bmatrix}$ is a sub matrix of A obtained by deleting third column from A.

Similarly $\begin{bmatrix} 1 & 3 & 8 \\ 9 & 8 & 7 \end{bmatrix}$ is a sub matrix of A obtained by deleting third row and 3^{rd}, 4^{th} column from A.

Minor of a matrix :—

Let A be an $m \times n$ matrix. The determinant of a square sub matrix of A is called a minor of the matrix. If the order of the square sub matrix is t then its determinant is called a minor of order t.

Eg :- $A = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 5 & 8 \\ 6 & 0 & 1 \end{bmatrix}_{4\times3}$ be a matrix.

We have $B = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$ be a sub matrix of order 2.

$|B| = 9 - 21 = -12$ is a minor of order 2.

Rank of a Matrix :—

Let A be an $m \times n$ matrix. If A is a null matrix. We define it Rank to be zero. If A is not null matrix. We say that r is the rank of A :

if (i) Every $(r+1)^{th}$ order minor of A is zero.

(ii) There exists atleast one r^{th} order minor of A which is not zero

Rank of A is denoted by $\rho(A)$.

Note :- (1) It can be noted that the rank of a non zero matrix is the order of the highest order non zero minor of A.

(2) Rank of a matrix is unique.

(3) Every matrix will have a rank.

(4) If A is a matrix of order $m \times n$ then

$$\text{Rank of } A = \rho(A) \leq \min\{m, n\}$$

Eg :- $A = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 12 \end{bmatrix}_{2\times 3}$

Given matrix is of order 2×3.

$$\rho(A) \leq \min\{2, 3\}$$

$$\text{i.e} \quad \rho(A) \leq 2.$$

$\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$ be sub matrix of order 2 of the given matrix.

$$\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 9 - 21 = -12 \neq 0$$

$$\therefore \rho(A) = 2.$$

(5) If $\rho(A) = r$ then every minor of A of order $r+1$ or more is zero.

Eg:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 1(2-0) - 2(4-0) + 3(2-0) = 0$$

$|A| = 0$ i.e A is singular

$\implies \rho(A) < 3$

Consider the minor of order 2, $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1-4 = -3 \neq 0.$

$\therefore \rho(A) = 2$.

(6) Rank of the identity matrix I_n is n

Eg:- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\rho(I) = 2$.

(7) If A is non singular matrix of order n then $\rho(A) = n$.

Eg:- $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 2 - 12 = -10 \neq 0$$

$\therefore |A| \neq 0$ i.e A is non singular

$\therefore \rho(A) = 2$.

(8) If A is a matrix, A^T is transpose of matrix A Then $\rho(A) = \rho(A^T)$

Eg:- $A = \begin{bmatrix} 1 & 2 & -5 \\ -3 & 4 & 6 \end{bmatrix}$

A is rectangular matrix of order 2×3.

$\rho(A) \leq \min\{2, 3\}$

$\rho(A) \leq 2$

Consider the minor of order 2, $\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} = 4+6 = 10 \neq 0$

$\therefore \rho(A) = 2$.

$$A^T = \begin{bmatrix} 1 & -3 \\ 2 & 4 \\ -5 & 6 \end{bmatrix}$$

A^T is rectangular matrix of order 3×2.

$\rho(A^T) \leq \min\{3, 2\}$

$\rho(A^T) \leq 2$

Consider the minor of order 2, $\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} = 4+6 = 10 \neq 0$.

$\therefore \rho(A) = 2$.

$\therefore \rho(A) = \rho(A^T)$

(9) If A is singular matrix of order n then $\rho(A) < n$.

Eg:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4 = 0$.

A is singular matrix

$\rho(A) < 2$.

A is not null matrix

$\therefore \rho(A) = 1$

(10) The Rank of non zero row matrix is 1.

Eg:- $A = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \end{bmatrix}_{1 \times 5}$

$\rho(A) = 1$.

(11) The Rank of non zero column matrix is 1.

Eg:- $A = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}_{4 \times 1}$ $\rho(A) = 1$.

(12) The rank of matrix is $\geq r$ if there is atleast one minor of r^{th} order which is not equal to zero.

→ Find the value of K such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & K & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2.

Sol: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & K & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Given that $\rho(A) = 2$.

So every minor of order greater than 2 is zero

i.e $|A| = 0$ i.e $\begin{vmatrix} 1 & 2 & 7 \\ 2 & K & 7 \\ 3 & 6 & 10 \end{vmatrix} = 0$

$1(10K - 42) - 2(20 - 21) + 7(12 - 3K) = 0$

$\therefore K = 4$

→ Find the rank of a matrix $A = \begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{bmatrix}_{3\times 4}$

A is rectangular matrix of order 3×4

$\rho(A) \leq \min\{3, 4\}$

$\rho(A) \leq 3$.

Consider the minor of order 3, $\begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} =$

$= 0 - 1(0 - 3) - 3(1) = 0$.

Consider the minor of order 3, $\begin{vmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} =$

$1(2 - 0) + 3(0 - 1) + 1(0 - 1) = -2 \neq 0$.

One minor of order 3 is not zero

$\therefore \rho(A) = 3$.

→ Find the rank of matrix $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

sol: G|T $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

A is a square matrix of order 3

$\rho(A) \leq 3$

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{vmatrix} = 3(4-4) + 1(-12+12) + 2(-6+6) = 0$$

Consider the minor of order 2, $\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} = -4-4 = -8 \neq 0$.

one minor of order 2 is not equal to zero.

$\therefore \rho(A) = 2$.

→ Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

sol: G|T $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

A is a square matrix of order 3

$\rho(A) \leq 3$.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{vmatrix} = 1(48-40) - 2(36-28) + 3(30-28)$$

$= 8 - 16 + 6 = -2 \neq 0$

$|A| \neq 0$

$\therefore \rho(A) = 3$.

→ Find the values of x suchthat matrix $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singu-lar.

Sol:- Given that $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & 4 & -1-x \end{bmatrix}$

A is singular $\Rightarrow |A| = 0$

$$\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & 4 & -1-x \end{vmatrix} = 0$$

$R_2 \longrightarrow R_2 + R_3$

$$\begin{vmatrix} 3-x & 2 & 2 \\ 0 & -x & -x \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$$-x \begin{vmatrix} 3-x & 2 & 2 \\ 0 & 1 & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$C_3 \longrightarrow C_3 - C_2$

$$-x \begin{vmatrix} 3-x & 2 & 0 \\ 0 & 1 & 0 \\ -2 & -4 & -3-x \end{vmatrix} = 0$$

Expand it by using 3rd column

$$x(3-x)^2 = 0$$

$$x = 0, 3.$$

Elementary transformations or operations on a matrix :-

(a) There are three types of elementary row operations.

(i) Interchange of two rows :- If i^{th} row and j^{th} row are interchanged, it is denoted by $R_i \leftrightarrow R_j$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 7 \\ 4 & 6 & 3 \\ 2 & 5 & -3 \end{bmatrix}$$

(ii) Multiplication of each element of a row with non zero scalar :-

If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow R_i(k)$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$R_2 \longrightarrow 2R_2$

$$\begin{bmatrix} 1 & 0 & 7 \\ 4 & 10 & -6 \\ 4 & 6 & 3 \end{bmatrix}$$

(iii) Multiplying every element of a row which is a non zero scalar and adding to the corresponding elements of another row :-

If the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by

$R_j \rightarrow R_j + kR_i$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$R_2 \longrightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 5 & -17 \\ 4 & 6 & 3 \end{bmatrix}$$

(b) There are three types of elementary column operations.

(i) Interchange of two columns :- If i^{th} column and j^{th} column are interchanged, it is denoted by $C_i \leftrightarrow C_j$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$\sim \begin{bmatrix} 0 & 1 & 7 \\ 5 & 2 & -3 \\ 6 & 4 & 3 \end{bmatrix}$

(ii) Multiplication of each element of a column with a non zero scalar :- If i^{th} row is multiplied with k then it is denoted by $C_i \rightarrow C_i(K)$.

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$C_2 \rightarrow C_2(2)$

$\sim \begin{bmatrix} 1 & 0 & 7 \\ 2 & 10 & -3 \\ 4 & 12 & 3 \end{bmatrix}$

(iii) Multiplying every element of a column which is a non zero scalar and adding to the corresponding elements of another column :- If the elements of i^{th} column are multiplied with k and added to the corresponding elements of j^{th} column then it is denoted by.

$C_j \rightarrow C_j + KC_i$

Eg :- $A = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & -3 \\ 4 & 6 & 3 \end{bmatrix}$

$C_3 \rightarrow C_3 - 7C_1$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & -17 \\ 4 & 6 & -25 \end{bmatrix}$

Equivalence of Matrices :-

If a matrix B is obtained from a matrix A after a finite chain of elementary transformations then B is said to be equivalent to A.

Symbolically it is denoted as $B \sim A$

Eg:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 5 & 9 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 5 & 9 \end{bmatrix} = B$$

Matrix B obtained from a matrix A after elementary row transformation. So the matrix B is said to be equivalent to A.

Zero row and Non zero row :-

If all the elements in a row of a matrix are zero's then it is called zero row and if there is atleast one zero element in a row then it is called a non zero row.

Eg:-
$$\begin{array}{l} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right. \\ \end{array}
\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 9 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\begin{array}{l} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \rightarrow \text{Non zero row} \\ \} \rightarrow \text{zero row.} \end{array}$$

Echelon form of a matrix :-

A Matrix is said to be Echelon form if the following three properties are satisfied.

(i) zero rows if any must be below the non zero rows.

(ii) The first non zero element of a non zero row is equal to one

(iii) The no. of zeros before the non zero element of a row is less than such zeros in the next row.

Note :- The condition (ii) is not compulsory.

Result :- The no. of non zero rows in a echelon form of A is the rank of A.

Eg :- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Is in echelon form since it satisfies all the three conditions of the echelon form

$\therefore \rho(A) = 3 =$ No. of non zero rows.

Working procedure to reduce a matrix into echelon form :-

Case (i) :-

Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

Step 1 :- If $a_{11} \neq 0$, by using a_{11} position, make a_{21} and a_{31} positions as zero. Here we apply row operations only.

$$\sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \end{bmatrix}$$

Step 2 :- If $a'_{22} \neq 0$, by using a'_{22} position, make a'_{32} position as zero. Here we apply row operations only.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \end{bmatrix}$$

Which is in echelon form.

$\rho(A) = 3$ if $a''_{33} \neq 0$ or $a''_{34} \neq 0$.

(or) $\rho(A) = 2$ if $a''_{33} = 0$ and $a''_{34} = 0$.

Case (ii) :- Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

Step 1 :- If $a_{11} \neq 0$, by using a_{11} position make a_{21}, a_{31} and a_{41} positions as zero. Here we apply row operations only.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix}$$

Step 2 :- If $a'_{22} \neq 0$, by using a'_{22} position make a'_{32} and a'_{42} positions as zero. Here we apply row operations only.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & a''_{43} & a''_{44} \end{bmatrix}$$

Step 3 :- If $a''_{33} \neq 0$ by using a''_{33} position make a''_{43} position as zero. Here we apply row operation only.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix}$$

Which is in echelon form

$\rho(A) = 4$ if $a'''_{44} \neq 0$

(or) $\rho(A) = 3$ if $a'''_{44} = 0$.

→ Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ by reduce it to echelon form.

Sol:- Given that $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Now we reduce the matrix A into echelon form by applying elementary row operations only.

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 2R_1, \quad R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

$R_4 \to 3R_4 - 2R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Which is in echelon form.

$\rho(A)$ = No. of non zero rows of the last equivalent to A $= 4$

$\therefore \rho(A) = 4$.

→ Show that the equations $x-3y-8z=-10$, $3x+y-4z=0$, $2x+5y+6z=13$ are consistent and solve the same.

Sol: Given that $x-3y-8z=-10$, $3x+y-4z=0$, $2x+5y+6z=13$

These are $m=3$ eqn's in $n=3$ unknowns x, y, z.

The matrix equation of the given system of eqn's is $AX=B$.

Where $A=\begin{bmatrix}1 & -3 & -8\\ 3 & 1 & -4\\ 2 & 5 & 6\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ $B=\begin{bmatrix}-10\\ 0\\ 13\end{bmatrix}$

The augmented matrix $[A|B]=\left[\begin{array}{ccc|c}1 & -3 & -8 & -10\\ 3 & 1 & -4 & 0\\ 2 & 5 & 6 & 13\end{array}\right]$

Now reduce the augmented matrix $[A|B]$ to echelon form by using E-row operations only and determine the $\rho(A)$ and $\rho([A|B])$ respectively.

$R_2 \to R_2-3R_1$, $R_3 \to R_3-2R_1$

$$\sim\left[\begin{array}{ccc|c}1 & -3 & -8 & -10\\ 0 & 10 & 20 & 30\\ 0 & 11 & 22 & 33\end{array}\right]$$

$R_3 \to 10R_3-11R_2$

$$\sim\left[\begin{array}{ccc|c}1 & -3 & -8 & -10\\ 0 & 10 & 20 & 30\\ 0 & 0 & 0 & 0\end{array}\right]$$

Which is in echelon form.

Hence $\rho(A)=2=$ The no. of non zero rows of equivalent A.

$\rho([A|B])=2=$ The no. of non zero rows of equivalent to $[A|B]$

$\rho(A)=\rho([A|B])=2<3$ (No. of unknowns)

Sothat the system is consistent and possesses an infinite no. of sol's.

To determine these solutions we have to assign arbitrary values

→ Discuss for what values of λ, μ the simultaneous equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (i) No solution. (ii) a unique solution (iii) an infinite no. of solutions.

Sol:- Given that $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$.

There are $m=3$ equations in $n=3$ unknowns x, y and z.

The matrix form of the given system of equations is $AX=B$.

Where $A=$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The augmented matrix $[A|B]=$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

Now reduce the augmented matrix $[A|B]$ to echelon form by using E-row operations only and determine ranks of A and $[A|B]$ respectively.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

Which is in echelon form.

Case(i):- No solution

Suppose $\lambda=3$ and $\mu\neq 0$. then $\rho(A)=2$ and $\rho([A|B])=3$

$\rho(A)\neq\rho([A|B])$

$\therefore$ The system is inconsistent

$\therefore$ It has no solution.

Now the equivalent matrix eqn. of $AX = B$ is .

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -8 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\beta \\ \tan\gamma \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

The corresponding system of eqns is

$$2\sin\alpha - \cos\beta + 3\tan\gamma = 3$$

$$4\cos\beta - 8\tan\gamma = -4$$

$$-8\tan\gamma = 0. \Rightarrow \gamma = 0$$

$$\cos\beta = \frac{-4 + \tan\gamma}{4}$$

$$\cos\beta = -1 \Rightarrow \beta = \pi$$

$$\sin\alpha = \frac{3 + \cos\beta - 3\tan\gamma}{2}$$

$$\sin\alpha = 1 \Rightarrow \alpha = \frac{\pi}{2}.$$

Hence $\alpha = \frac{\pi}{2}$, $\beta = \pi$ and $\gamma = 0$ is the sol. of the system.

SYSTEM OF NON HOMOGENEOUS LINEAR EQUATIONS

5

1. Test for consistency and hence solve $x+y+z=6$, $x-y+2z=5$, $3x+y+z=8$, $2x-2y+3z=7$ Ans:- $x=1$ $y=2$ $z=3$.

2. Test for consistency $3x+3y+2z=1$, $x+2y-4=0$, $10y+3z=-2$, $2x-3y-z=5$ Ans:- $x=2$ $y=1$ $z=-4$.

3. If consistent, solve $x+y+z+t=4$, $x-z+2t=2$, $y+z-3t=-1$, $x+2y-z+t=3$.
   Ans:- $x=y=z=t=1$.

4. Solve completely the equations $3x-2y-w=2$, $2y+2z+w=1$, $y+2z+w=1$, $x-2y-3z+2w=3$ Ans:- $x=w=1$, $y=z=0$.

5. Show that the equations $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$, $x-y+z=-1$ are consistent and solve them $x=-1$, $y=4$, $z=4$.

6. Solve the system for x, y and z, $-\frac{1}{x}+\frac{3}{y}+\frac{4}{z}=30$, $\frac{3}{x}+\frac{2}{y}+\frac{1}{z}=9$, and $\frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10$. Ans: $x=\frac{1}{2}$, $y=\frac{1}{4}$, $z=\frac{1}{5}$.

7. Solve the following system of non linear equations for the unknown angles α, β and γ where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$ and $0 \le \gamma < \pi$.
   $2\sin\alpha - \cos\beta + 3\tan\gamma = 3$, $4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2$, $6\sin\alpha - 3\cos\beta + \tan\gamma = 9$.
   Ans:- $\alpha = \frac{\pi}{2}$, $\beta = \pi$, $\gamma = 0$.

8. Determine the values of λ for which the system $3x-y+\lambda z=0$, $2x+y+z=2$, $x-2y-\lambda z=-1$ will fail to have a unique solution. For what value of λ are the equations consistent. Ans:- $\lambda = -\frac{7}{2}$, No solution.

9. For what values of a and b the equations $x+2y+3z=8$, $2x+y+3z=13$, $3x+4y-az=b$ have (i) No solution (ii) A unique solution (iii) An infinite no. of solutions.

10. Solve the system if consistent $x+y+z=-3$, $3x+y-2z=-2$, $2x+4y+7z=7$ are inconsistent.

SYSTEM OF NON HOMOGENEOUS LINEAR EQUATIONS.

1 Are the following equations consistent, if so solve them.

$x_1 - x_2 + x_3 - x_4 + x_5 = 1$, $2x_1 - x_2 + 3x_3 + 4x_5 = 2$, $3x_1 - 2x_2 + 2x_3 + x_4 + x_5 = 1$.

$x_1 + x_3 + 2x_4 + x_5 = 0$. Ans:- $x_4 = k_1$ $x_5 = k_2$, $x_3 = 1 + k_1 - 2k_2$,

$x_2 = -1 - 3k_1$, $x_1 = -1 + 3k_1 + k_2$.

2 Solve the system completely $x+y+z=1$, $x+2y+4z=\alpha$, $x+4y+10z=\alpha^2$.

Ans:- $\alpha=1$, $x = 1+2k_1$, $y=-3k_1$, $z=k_1$; $\alpha=2$, $x=2k_2$, $y=1-3k_2$, $z=k_2$.

3 show that the equations $-2x+y+z=a$, $x-2y+z=b$, $x+y-2z=c$ have no solution unless $a+b+c=0$, in which case, they have infinitely many solution. Find these solutions when $a=1$, $b=1$, $c=-2$.

Ans:- $x=k-1$, $y=k-1$, $z=k$.

4 Find for what values of λ, the set of equations $2x-3y+6z-5t=3$, $y-4z+t=1$, $4x-5y+8z-9t=\lambda$ has (i) No solution (ii) Infinite number of solutions and find the solution of the equations when they are consistent.

Ans:- (i) $\lambda \neq 7$ (ii) $\lambda=7$, $x = 3k_1+k_2+3$, $y = 4k_1-k_2+1$, $z=k_1$, $t=k_2$

5 Show that if $\lambda \neq 0$, the system of equations $2x+y=a$, $x+\lambda y-z=b$, $y+2z=c$ has a unique solution for every value of a, b, c. If $\lambda=0$, determine the relation satisfied by a, b, c such that the system is consistent. Find the solution by taking $\lambda=0$, $a=1$, $b=1$, $c=-1$. Ans:- $x=1+k_1$, $y=-1-2k_1$, $z=k_1$

6 Find the value of λ for which the system of equations $3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$ will have infinite number of solutions and solve them with the same λ value. Ans:- $x = \frac{4-5k}{7}$, $y = \frac{13k-9}{7}$, $z=k$.

7 show that the equations $4x-y+6z=16$, $x-4y-3z=-16$, $2x+7y+12z=48$, $5x-5y+3z=0$ are consistent and solve the same Ans:- $z=k$, $y=\frac{16}{3}-\frac{16}{5}k$,

$x = \frac{16}{3} - \frac{9}{5}k$.

8 Solve $u+2v+2w=1$, $2u+v+w=2$, $3u+2v+2w=3$, $v+w=0$ Ans:- $u=1$, $v=-c$, $w=c$.

Method of Factorization [L-U De Composition Method]:

(Triangularisation):-

This method is based on the fact that a square matrix A can be factorized into the form LU where L is the unit lower triangular matrix and U is the upper triangular matrix. Here all principal minors of A must be non singular. This factorisation if it exists, is unique.

Consider a system of linear equations
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Which can be written in the matrix form $AX = B$ — ①.

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Let $A = LU$ — ②.

Where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ — ③ is the unit lower triangular matrix.

$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ is an upper triangular matrix.

Then from ① and ② $LUX = B$ — ③

Put $UX = Y$ — ④ Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then ③ can be written as $LY = B$ — ⑤.

$$⑤ \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$$y_1 = b_1$$
$$l_{21}y_1 + y_2 = b_2$$
$$l_{31}y_1 + l_{32}y_2 + y_3 = b_3$$

This can be solved for y_1, y_2, y_3 by forward substitution.

Then (4) $\Rightarrow$ $UX = Y$.

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3.$$

Which can be solved for x_1, x_2, and x_3 by backward substitution.

Computation of Lower and Upper triangular matrices L and U :—

From equation (2) we have

$$LU = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now equating the corresponding elements on both sides, we get

$$u_{11} = a_{11} \qquad u_{12} = a_{12} \qquad u_{13} = a_{13}$$

$$l_{21}u_{11} = a_{21} \implies l_{21} = \frac{a_{21}}{u_{11}} = \frac{a_{21}}{a_{11}}.$$

$$l_{31}u_{11} = a_{31} \implies l_{31} = \frac{a_{31}}{u_{11}} = \frac{a_{31}}{a_{11}}$$

$$l_{21}u_{12} + u_{22} = a_{22} \implies u_{22} = a_{22} - l_{21}u_{12}$$

$$u_{22} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}$$

$$l_{21}u_{13} + u_{23} = a_{23} \implies u_{23} = a_{23} - l_{21}u_{13}$$

$$u_{23} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$$

$$l_{31}u_{12} + l_{32}u_{22} = a_{32} \implies l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}$$

$$l_{32} = \frac{a_{32} - \left(\frac{a_{31}}{a_{11}}\right)a_{12}}{a_{22} - \left(\frac{a_{21}}{a_{11}}\right)a_{12}}$$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$ from which u_{33} can be calculated.

We have a systematic procedure to evaluate the elements of L and U.

Step 1 :- We determine the first row of U and the first column of L.

Step 2 :- We determine the second row of U and the second column of L

Step 3 :- Finally we compute the third row of U. This procedure can be obviously generalized. This method is also called as L-U decomposition method.

(1) Solve the system of equations $2x+3y+z=9$, $x+2y+3z=6$, $3x+y+2z=8$ by the factorization method.

Sol:- Given that

$$\left.\begin{aligned} 2x+3y+z&=9\\ x+2y+3z&=6\\ 3x+y+2z&=8 \end{aligned}\right\} \text{———} ①$$

The matrix form of the given system of eqns is $AX=B$ ——— ②

where $A=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}$ $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ $B=\begin{bmatrix}9\\6\\8\end{bmatrix}$

Let $A=LU$ ——— ②

where $L=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}$ is the unit lower triangular matrix

$U=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$ is the upper triangular matrix.

From ① and ②, $LUX=B$ ——— ③

Taking $UX=Y$ ——— ④ where $Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

From ③ and ④, $LY=B$ ——— ⑤

To find the matrices L and U :—

From equation ② we have $LU=A$

$$\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}2&3&1\\1&2&3\\3&1&2\end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Equating the corresponding elements bothsides, we get

$u_{11} = 2 \qquad u_{12} = 3 \qquad u_{13} = 1$

$l_{21}u_{11} = 1 \implies l_{21} = \frac{1}{2}$

$l_{31}u_{11} = 3 \implies l_{31} = \frac{3}{2}$.

$l_{21}u_{12} + u_{22} = 2 \implies \frac{3}{2} + u_{22} = 2$ i.e $u_{22} = 2 - \frac{3}{2} = \frac{1}{2}$.

$l_{21}u_{13} + u_{23} = 3 \implies \frac{1}{2} + u_{23} = 3$ i.e $u_{23} = 3 - \frac{1}{2} = \frac{5}{2}$.

$l_{31}u_{12} + l_{32}u_{22} = 1 \implies \frac{9}{2} + l_{32}\frac{1}{2} = 1$ i.e $l_{32} = -7$.

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$

$$\implies \frac{3}{2} - \frac{35}{2} + u_{33} = 2$$

$$u_{33} = 2 - \frac{3}{2} + \frac{35}{2}$$

$$u_{33} = 18$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

From equation (5), First we have to find the values of y_1, y_2 and y_3.

i.e $LY = B \implies \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

$$y_1 = 9$$

$$\frac{1}{2}y_1 + y_2 = 6$$

$$\frac{3}{2}y_1 - 7y_2 + y_3 = 8$$

solving the above equations by forward substitution.

$$y_2 = 6 - \frac{1}{2}y_1 = 6 - \frac{1}{2}9 = \frac{3}{2}$$

$$y_3 = 8 - \frac{3}{2}y_1 + 7y_2 = 8 - \frac{27}{2} + \frac{21}{2}$$

$$y_3 = 5$$

$$\therefore\ y_1 = 6 \quad y_2 = \frac{3}{2} \quad y_3 = 5.$$

From the equation (4), we have to find the values of x, y and z.

$$UX = Y \implies \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$$2x + 3y + z = 9$$

$$\frac{1}{2}y + \frac{5}{2}z = \frac{3}{2}$$

$$18z = 5$$

solving the above eqn's by backward substitution.

$$z = \frac{5}{18}$$

$$\frac{5}{2}z = \frac{3}{2} - \frac{1}{2}y \quad (\text{OR}) \quad \frac{y}{2} = \frac{3}{2} - \frac{5z}{2}$$

$$\frac{y}{2} = \frac{3}{2} - \frac{5}{2}\cdot\frac{5}{18}$$

$$y = 3 - \frac{25}{18} = \frac{29}{18}$$

$$z = 9 - 2x - 3y \quad (\text{OR}) \quad 2x = 9 - 3y - z$$

$$2x = 9 - 3\cdot\frac{29}{18} - \frac{5}{18} = \frac{70}{18}$$

$$x = \frac{35}{18}.$$

$\therefore$ The solution of the given system is $x = \frac{35}{18}$, $y = \frac{29}{18}$, $z = \frac{5}{18}$.

→ Solve the system $x+2y+3z=10$, $3x+y+2z=13$, $2x+3y+z=13$ by LU Decomposition Method.

Sol:- Given that $x+2y+3z=10$, $3x+y+2z=13$, $2x+3y+z=13$

The matrix form of the given system of eqn's is $AX=B$ — (1)

Where $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 2\\ 2 & 3 & 1\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ $B=\begin{bmatrix}10\\ 13\\ 13\end{bmatrix}$

Step (i) :- Let $A=LU$ —— (2)

where $L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$ is the unit lower triangular matrix

$U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$ is the upper triangular matrix.

From (1) and (2), $LUX=B$ —— (3)

Taking $UX=Y$ —— (4) where $Y=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$

From (3) and (4), $LY=B$ —— (5)

Step (ii) :- To find the matrices L and U :-

From equation (2), we have $LU=A$

$$\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 3 & 1\end{bmatrix}$$

$$\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 2\\ 2 & 3 & 1\end{bmatrix}$$

Equating the corresponding elements bothsides, we get

$$u_{11} = 1 \qquad u_{12} = 2 \qquad u_{13} = 3.$$

$$l_{21} u_{11} = 3 \implies l_{21} = 3.$$

$$l_{31} u_{11} = 2 \implies l_{31} = 2$$

$$l_{21} u_{12} + u_{22} = 1 \implies u_{22} = 1 - l_{21} u_{12}$$

$$u_{22} = 1 - 3(2) = -5$$

$$l_{21} u_{13} + u_{23} = 2 \implies u_{23} = 2 - l_{21} u_{13}$$

$$u_{23} = 2 - 3(3) = -7.$$

$$l_{31} u_{12} + l_{32} u_{22} = 3 \implies l_{32} = \frac{3 - l_{31} u_{12}}{u_{22}} = \frac{3-4}{-5} = \frac{1}{5}$$

$$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 1 \implies u_{33} = 1 - l_{31} u_{13} - l_{32} u_{23}$$

$$u_{33} = 1 - 6 + \frac{7}{5} = \frac{-18}{5}$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{5} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & \frac{-18}{5} \end{bmatrix}$$

Step(iii) :- From equation (5) first we have to find the values of y_1, y_2 and y_3.

i.e $LY = B \implies \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 13 \end{bmatrix}$

$$y_1 = 10$$

$$3y_1 + y_2 = 13$$

$$y_2 = 13 - 3y_1 = 13 - 30$$

$$y_2 = -17.$$

$$2y_1 + \frac{1}{5} y_2 + y_3 = 13.$$

$$y_3 = 13 - 2y_1 - \frac{1}{5} y_2$$

$$y_3 = \frac{-18}{5}$$

$$\therefore Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -17 \\ \frac{-18}{5} \end{bmatrix}$$

step (iv) :- From the equation ④ we have to find the values of x, y and z.

$$UX = Y \implies \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & \frac{-18}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -17 \\ \frac{-18}{5} \end{bmatrix}$$

$$x + 2y + 3z = 10$$

$$-5y - 7z = -17 \implies 5y + 7z = 17$$

$$\frac{-18}{5} z = \frac{-18}{5} \implies z = 1.$$

$$y = \frac{17 - 7z}{5} = \frac{17 - 7}{5} = 2$$

$$x = 10 - 2y - 3z$$

$$x = 10 - 4 - 3$$

$$x = 3.$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$ is the solution of the given system.

→ Solve $-3x+12y-6z=-33$; $x-2y+2z=7$, $y+z=-1$ using LU-Decomposition Method.

Sol:- Given that $-3x+12y-6z=-33$, $x-2y+2z=7$, $y+z=-1$

The matrix form of the given system is $AX=B$ —— ①.

where $A=\begin{bmatrix}-3 & 12 & -6\\ 1 & -2 & 2\\ 0 & 1 & 1\end{bmatrix}$ $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ $B=\begin{bmatrix}-33\\ 7\\ -1\end{bmatrix}$

<u>Step (i)</u> :- Let $A=LU$ —— ②

Where $L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$ is the unit lower triangular matrix

$U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$ is the upper triangular matrix.

From ① and ②, $LUX=B$ —— ③

Taking $UX=Y$ —— ④ where $Y=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$

From ③ and ④, $LY=B$ —— ⑤

<u>Step (ii)</u> :- <u>To find the matrices L and U</u> :—

From equation ②, we have $LU=A$

$$\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}-3 & 12 & -6\\ 1 & -2 & 2\\ 0 & 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}-3 & 12 & -6\\ 1 & -2 & 2\\ 0 & 1 & 1\end{bmatrix}$$

Equating the corresponding elements bothsides, we get

$$u_{11} = -3 \qquad u_{12} = 12 \qquad u_{13} = -6 .$$

$$l_{21} u_{11} = 1 \implies l_{21} = -\frac{1}{3}.$$

$$l_{31} u_{11} = 0 \implies l_{31} = 0 .$$

$$l_{21} u_{12} + u_{22} = -2 \implies u_{22} = -2 - l_{21} u_{12}$$
$$u_{22} = -2 + \tfrac{1}{3}(12) = 2.$$

$$l_{21} u_{13} + u_{23} = 2 \implies u_{23} = 2 - l_{21} u_{13} = 2 - \left(-\tfrac{1}{3}\right)(-6)$$
$$u_{23} = 0 .$$

$$l_{31} u_{12} + l_{32} u_{22} = 1 \implies l_{32} = \frac{1 - l_{31} u_{12}}{u_{22}} = \frac{1-0}{2} = \frac{1}{2}.$$

$$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 1 \implies u_{33} = 1 - l_{31} u_{13} - l_{32} u_{23}$$
$$u_{33} = 1 - 0 = 1.$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} -3 & 12 & -6 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step (iii) :- From equation (5) first we have to find the values of y_1, y_2 and y_3.

$$\text{i.e } LY = B \implies \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$y_1 = -33$$
$$-\tfrac{1}{3} y_1 + y_2 = 7$$
$$\tfrac{1}{2} y_2 + y_3 = -1.$$

$$y_2 = 7 + \frac{1}{3}y_1 = 7 + \frac{1}{3}(-33) = -4$$

$$y_3 = -1 - \frac{1}{2}y_2 = -1 - \frac{1}{2}(-4) = 1.$$

$$\therefore \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ -4 \\ 1 \end{bmatrix}.$$

Step (iv) :- From the equation (4), we have to find the values of x, y and z

$$UX = Y \implies \begin{bmatrix} -3 & 12 & -6 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -33 \\ -4 \\ 1 \end{bmatrix}$$

$$-3x + 12y - 6z = -33$$

$$2y = -4 \implies y = -2$$

$$z = 1.$$

$$3x = 33 + 12y - 6z$$

$$x = \frac{33 + 12y - 6z}{3}$$

$$x = \frac{33 - 24 - 6}{3} = 1.$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$ is the solution of the given system.

CROUT'S METHOD :-

Consider the linear system
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{---} (1).$$

Which can be written in the matrix form $AX = B$ — (2).

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let $A = LU$ — (3)

Where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Here L is the Lower triangular ~~for~~ matrix.

U is the unit upper triangular matrix.

Then from (2) and (3), $LUX = B$ — (4).

Put $UX = Y$ — (5) Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Then (4) can be written as $LY = B$ — (6).

$$(6) \Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$l_{11}y_1 = b_1$$

$$l_{21}y_1 + l_{22}y_2 = b_2$$

$$l_{31}y_1 + l_{32}y_2 + l_{33}y_3 = b_3.$$

This can be solved for y_1, y_2, y_3 by forward substitution.

Then (5) $\Rightarrow UX = Y$

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$$x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$x_2 + u_{23}x_3 = y_2$$
$$x_3 = y_3$$

Which can be solved for x_1, x_2, x_3 and by backward substitution.

Computation of Lower and Upper triangular Matrices :—

We have. $LU = A$.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{23} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now equating the corresponding elements on bothsides, we get

$$l_{11} = a_{11} \qquad l_{11}u_{12} = a_{12} \qquad l_{11}u_{13} = a_{13}$$
$$l_{21} = a_{21} \qquad l_{21}u_{12} + l_{22} = a_{22} \qquad l_{21}u_{23} + l_{22}u_{23} = a_{23}$$
$$l_{31} = a_{31} \qquad l_{31}u_{12} + l_{32} = a_{32} \qquad l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}$$

From this, we obtain $u_{12}, u_{13}, u_{23}, l_{22}, l_{32}, l_{33}$ and thus L and U are obtained.

Use crout's method to solve the system $x+y+z=1$ $3x+y-3z=5$ $x-2y-5z=10$.

Sol:- Given that $x+y+z=1$ $3x+y-3z=5$ $x-2y-5z=10$.

The matrix form of the given system is $AX=B$ —— (1)

Where $A=\begin{bmatrix}1&1&1\\3&1&-3\\1&-2&-5\end{bmatrix}$ $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ $B=\begin{bmatrix}1\\5\\10\end{bmatrix}$

Let $A=LU$ —— (2).

Where $L=\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}$ is the lower triangular matrix.

$U=\begin{bmatrix}1&u_{12}&u_{13}\\0&1&u_{23}\\0&0&1\end{bmatrix}$ is the unit upper triangular matrix.

From (1) and (2), $LUX=B$ —— (3).

Taking $UX=Y$ —— (4) where $Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$

From (3) and (4), $LY=B$ —— (5).

To find the matrices L and U :—

From equation (2), we have $LU=A$.

$$\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}\begin{bmatrix}1&u_{12}&u_{13}\\0&1&u_{23}\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&1\\3&1&-3\\1&-2&-5\end{bmatrix}$$

$$\begin{bmatrix}l_{11}&l_{11}u_{12}&l_{11}u_{13}\\l_{21}&l_{21}u_{12}+l_{22}&l_{21}u_{13}+l_{22}u_{23}\\l_{31}&l_{31}u_{12}+l_{32}&l_{31}u_{13}+l_{32}u_{23}+l_{33}\end{bmatrix}=\begin{bmatrix}1&1&1\\3&1&-3\\1&-2&-5\end{bmatrix}$$

Equating the corresponding elements bothsides, we get

$l_{11}=1$

$l_{11}u_{12}=1 \Longrightarrow u_{12}=1.$

$l_{11}u_{13}=1 \Longrightarrow u_{13}=1.$

$l_{21} = 3.$

$l_{21}u_{12} + l_{22} = 1 \implies l_{22} = 1 - l_{21}u_{12}$

$l_{22} = 1 - 3.(1) = -2.$

$l_{21}u_{13} + l_{22}u_{23} = -3 \implies u_{23} = \dfrac{-3 - l_{21}u_{13}}{l_{22}}$

$u_{23} = \dfrac{-3 - 3(1)}{-2} = 3.$

$l_{31} = 1$

$l_{31}u_{12} + l_{32} = -2. \implies l_{32} = -2 - l_{31}u_{12}$

$= -2 - 1(1) = -3$

$l_{32} = -3.$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = -5$

$\implies l_{33} = -5 - l_{31}u_{13} - l_{32}u_{23}$

$l_{33} = -5 - 1(1) - (-3)(3) = -5 - 1 + 9 = 3.$

$l_{33} = 3.$

$$\therefore L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -3 & 3 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

From equation (5), First we have to find the values of y_1, y_2, and y_3.

i.e $LY = B \implies \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$

$y_1 = 1$

$3y_1 - 2y_2 = 5$

$y_1 - 3y_2 + 3y_3 = 10.$

solving the above equations by forward substitution.

$y_2 = \dfrac{3y_1 - 5}{2} \implies y_2 = -1.$

$$y_3 = \frac{10 + 3y_2 - y_1}{3}$$

$$y_3 = \frac{10 - 3 - 1}{3} = 2 .$$

From the equation (4), we have to find the values of x, y and z.

$$UX = Y \implies \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$x + y + z = 1$$
$$y + 3z = -1$$
$$z = 2$$

Solving the above equation by backward substitution.

$$z = 2$$

$$y = -1 - 3z = -7.$$

$$x = 1 - y - z = 1 + 7 - 2 = 6.$$

$\therefore x = 6, \quad y = -7 \quad z = 2 .$

Which is the required solution of the given system.

Solution to Tri-diagonal Systems :-

Definition :- If the coefficient matrix of a system of linear equations ie $AX = B$ has non zero elements along the main diagonal and the adjacent diagonals on either side of the main diagonal, then the system is called a "Tri diagonal system".

Working procedure :-

Consider the system of equations.

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3$$

$$a_{43}x_3 + a_{44}x_4 = b_4 .$$

Step 1 :- The matrix equation of the given tri diagonal system is

$$AX = B \quad \text{—— (1)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{3\text{[illegible]}} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{bmatrix}$ is the coefficient matrix of the system.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is the matrix of unknowns $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is the constant matrix.

Step 2 :- Let $A = LU$ —— (2).

Where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}$ is the unit lower triangular matrix.

$U = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$ is an upper triangular matrix.

From (1) and (2), $LUX = B$ —— (3).

Step 3 :- Put $UX = Y$ —④ Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

From ③ $LY = B$.

i.e $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

The linear equations are
$$y_1 = b_1$$
$$l_{21} y_1 + y_2 = b_2$$
$$l_{32} y_2 + y_3 = b_3$$
$$l_{43} y_3 + y_4 = b_4$$

This can be solved for y_1, y_2 and y_3, y_4 by forward substitution.

Step 4 :- Using ④ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$, we get

$$UX = Y \Rightarrow \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

The linear equations are
$$u_{11} x_1 + u_{12} x_2 = y_1$$
$$u_{22} x_2 + u_{23} x_3 = y_2$$
$$u_{33} x_3 + u_{34} x_4 = y_3$$
$$u_{44} x_4 = y_4$$

Which can be solved for x_1, x_2, x_3 and x_4 by backward substitution.

Thus when L and U are known, we can calculate $y_1, y_2, y_3\ y_4$ and x_1, x_2, x_3, x_4 by the above process.

Computation of L and U :-

We have $A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & u_{23} & 0 \\ 0 & l_{32}u_{22} & l_{32}u_{23}+u_{33} & u_{34} \\ 0 & 0 & l_{43}u_{33} & l_{43}u_{34}+u_{44} \end{bmatrix}$$

Equating the corresponding elements on bothsides.

$u_{11} = a_{11}$, $u_{12} = a_{12}$

$l_{21}u_{11} = a_{21} \implies l_{21} = \dfrac{a_{21}}{u_{11}}$, $l_{21}u_{12} + u_{22} = a_{22}$

$u_{22} = a_{22} - u_{12}l_{21}$

$u_{23} = a_{23}$, $l_{32}u_{22} = a_{32}$

$\implies l_{32} = \dfrac{a_{32}}{u_{22}}$

$l_{32}u_{23} + u_{33} = a_{33} \implies u_{33} = a_{33} - l_{32}u_{23}$.

$u_{34} = a_{34}$

$a_{34} = u_{33}l_{43} \implies l_{43} = \dfrac{a_{34}}{u_{33}}$.

$a_{44} = l_{43}u_{34} + u_{44} \implies u_{44} = a_{44} - l_{43}u_{34}$.

→ Solve the system of equations $2x-y=0$, $-x+2y-z=0$, $-y+2z-u=0$
$-z+2u=1$

Sol:- Given that
$$2x-y=0$$
$$-x+2y-z=0$$
$$-y+2z-u=0$$
$$-z+2u=1$$

The matrix equation of the given system of equations is $AX=B$ — (1)

Where $A=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ is the Tri-diagonal matrix.

$X=\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ and $B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Now we solve this system by L-U decomposition method or Method of factorization.

Let $A=LU$ — (2)

Where $L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}$ and $U=\begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$

From (1) and (2), $LUX=B$ — (3)

Taking $UX=Y$ — (4) Where $Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

From (3) and (4), $LY=B$.

$$LU=A \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & u_{23} & 0 \\ 0 & l_{32}u_{22} & l_{32}u_{23}+u_{33} & u_{34} \\ 0 & 0 & l_{43}u_{33} & l_{43}u_{34}+u_{44} \end{bmatrix}=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Equating the corresponding elements on both sides, we get

$u_{11} = 2, \; u_{12} = -1, \; u_{23} = -1, \; u_{34} = -1.$

$l_{21} u_{11} = -1 \implies l_{21} = \frac{-1}{u_{11}} = \frac{-1}{2}.$

$l_{21} u_{12} + u_{22} = 2 \implies u_{22} = 2 - l_{21} u_{12} = 2 - \left(\frac{-1}{2}\right)(-1) = \frac{3}{2}$

$l_{32} u_{22} = -1 \implies l_{32} = \frac{-1}{u_{22}} = -\frac{2}{3}.$

$l_{32} u_{23} + u_{33} = 2 \implies u_{33} = 2 - \left(-\frac{2}{3}\right)(-1) = \frac{4}{3}.$

$l_{43} u_{33} = -1 \implies l_{43} = \frac{-1}{u_{33}} = \frac{-3}{4}.$

$l_{43} u_{34} + u_{44} = 2 \implies u_{44} = 2 - \left(\frac{-3}{4}\right)(-1) = \frac{5}{4}.$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

From ⑤, first we have to find the values of y_1, y_2, y_3 and y_4.

$$\text{i.e } LY = B \implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ +1 \end{bmatrix}$$

Solving the system by forward substitution, we have $y_1 = 0$.

$$-\frac{1}{2} y_1 + y_2 = 0 \implies y_2 = 0$$

$$-2/3\, y_2 + y_3 = 0 \implies y_3 = 0.$$

$$-\frac{3}{4} y_3 + y_4 = 1 \implies y_4 = 1$$

$$\therefore Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Now from equation ④, we have to find the value of x, y, z and u.

$$UX = Y \Rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving the system by backward substitution, we have.

$$\frac{5}{4}u = 1 \Rightarrow u = \frac{4}{5}.$$

$$\frac{4}{3}z - u = 0 \Rightarrow \frac{4}{3}z = \frac{4}{5} \Rightarrow z = \frac{3}{5}.$$

$$\frac{3}{2}y - z = 0 \Rightarrow \frac{3}{2}y = z \Rightarrow \frac{3}{2}y = \frac{3}{5} \Rightarrow y = \frac{2}{5}.$$

$$2x - y = 0 \Rightarrow x = \frac{y}{2} = \frac{1}{5}.$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \\ 3/5 \\ 4/5 \end{bmatrix}$$ is the solution of the given system.

→ Solve the system of equations $2x_1 + x_2 = 2$, $x_1 + 2x_2 + x_3 = 2$, $x_2 + 2x_3 + x_4 = 2$, $x_3 + 2x_4 = 1$.

Sol:- Given that

$$2x_1 + x_2 = 2$$
$$x_1 + 2x_2 + x_3 = 2$$
$$x_2 + 2x_3 + x_4 = 2$$
$$x_3 + 2x_4 = 1.$$

The matrix equation of the given system of equations is $AX = B$ ——①.

Where $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ is the Tri diagonal matrix. $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

Now we solve this system by L-U decomposition method or Method of factorization.

Let $A = LU$ ——② Where $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix}$ $U = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$

From ① & ②, We write $LUX = B$ —— ③

Taking $UX = Y$ —— ④ Where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$

From ③ and ④, $LY = B$ —— ⑤

$$LU = A \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ 0 & l_{32} & 1 & 0 \\ 0 & 0 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ 0 & u_{22} & u_{23} & 0 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & 0 & 0 \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & u_{23} & 0 \\ 0 & l_{32}u_{22} & l_{32}u_{23}+u_{33} & u_{34} \\ 0 & 0 & l_{43}u_{33} & l_{43}u_{34}+u_{44} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Equating the corresponding elements on both sides, we get

$u_{11} = 2 \quad u_{12} = 1 \quad u_{23} = 1 \quad u_{34} = 1.$

$l_{21}u_{11} = 1 \Longrightarrow l_{21} = \frac{1}{u_{11}} = \frac{1}{2}$

$l_{21}u_{12} + u_{22} = 2 \Longrightarrow u_{22} = 2 - l_{21}u_{12} = 2 - \frac{1}{2}(1) = \frac{3}{2}.$

$l_{32}u_{22} = 1 \Longrightarrow l_{32} = \frac{1}{u_{22}} = \frac{2}{3}.$

$l_{32}u_{23} + u_{33} = 2 \Longrightarrow u_{33} = 2 - l_{32}u_{23} = 2 - \frac{2}{3}(1) = \frac{4}{3}.$

$l_{43}u_{33} = 1 \Longrightarrow l_{43} = \frac{1}{u_{33}} = \frac{3}{4}.$

$l_{43}u_{34} + u_{44} = 2 \Longrightarrow u_{44} = 2 - l_{43}u_{34} = 2 - \left(\frac{3}{4}\right)(1) = \frac{5}{4}.$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

Using equation ⑤, first we have to find the values of y_1, y_2, y_3 and y_4

$$LY = B \implies \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Solving the system by forward substitution, we have.

$y_1 = 2$

$\frac{1}{2} y_1 + y_2 = 2 \implies y_2 = 2 - \frac{1}{2}(2) = 1$

$\frac{2}{3} y_2 + y_3 = 2 \implies y_3 = 2 - \frac{2}{3} y_2 = 2 - \frac{2}{3}(1) = \frac{4}{3}.$

$\frac{3}{4} y_3 + y_4 = 1 \implies y_4 = 1 - \frac{3}{4} y_3 = 1 - \frac{3}{4}\left(\frac{4}{3}\right) = 0.$

$$\therefore \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4/3 \\ 0 \end{bmatrix}$$

Now using equation ④, we have to find the values of x_1, x_2, x_3 and x_4.

$$UX = Y \implies \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4/3 \\ 0 \end{bmatrix}$$

Solving the system by backward substitution, we have.

$\frac{5}{4} x_4 = 0 \implies x_4 = 0.$

$\frac{4}{3} x_3 + x_4 = \frac{4}{3} \implies x_3 = 1$

$\frac{3}{2} x_2 + x_3 = 1 \implies x_2 = 0$

$2x_1 + x_2 = 2 \implies x_1 = 1$

$$\therefore \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Note:- In the method of decomposition or in the method of solving tri-diagonal system, we can take L and U such that L is unit lower triangular & U is upper triangular (or) L is lower triangular and U is unit upper triangular

LU - DECOMPOSITION METHOD

1) Solve the system $x+y+z=1$, $3x+y-3z=5$, $x-2y-5z=10$ by using the LU decomposition method. Ans:- $x=6$, $y=-7$, $z=2$.

2) Solve the system $4x+y+z=4$, $x+4y-2z=4$, $3x+2y-4z=6$ by using Method of factorization. Ans:- $x=1$, $y=\frac{1}{2}$, $z=\frac{-1}{2}$.

3) Solve the system $x_1+3x_2+8x_3=4$, $x_1+4x_2+3x_3=-2$, $x_1+3x_2+4x_3=1$ by Triangularisation Method. Ans:- $x_1=\frac{19}{4}$ $x_2=\frac{-9}{4}$, $x_3=\frac{3}{4}$.

4) Solve the following matrix equation by using the LU-decomposition method.

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

Ans:- $x=1$, $y=-2$, $z=1$

5) Solve the system of equations $x+y+z=3$, $x+2y+3z=6$, $x+y+4z=6$ by using Triangularisation Method. Ans:- $x=y=z=1$.

6) Solve the system of equations $10x+y+2z=13$, $3x+10y+z=14$, $2x+3y+10z=15$ by using Method of factorization Ans:- $x=y=z=1$.

7) Solve the system of equations $x+y-z=2$, $2x+3y+5z=-3$, $3x+2y-3z=6$ by using LU decomposition method. Ans:- $x=1$, $y=0$, $z=-1$.

8) Solve the system of equations $2x+y+4z=12$, $4x+11y-z=33$, $8x-3y+2z=20$ by using LU decomposition method Ans:- $x=3$, $y=2$, $z=1$.

9) Solve the following equations by expressing the coefficient matrix as a product of a lower triangular and upper triangular matrices.
$2x+y-z=3$, $x-2y-2z=1$, $x+2y-3z=9$ Ans:- $x=-\frac{1}{5}$, $y=\frac{7}{5}$, $z=-2$

10 Solve the following equations using LU decomposition method.
$10x_1+7x_2+8x_3+7x_4=32$, $7x_1+5x_2+6x_3+5x_4=23$, $8x_1+6x_2+10x_3+9x_4=33$, $7x_1+5x_2+9x_3+10x_4=31$. Ans:- $x_1=x_2=x_3=x_4=1$.

SOLUTION OF TRIDIAGONAL SYSTEMS.

1 Solve the following tridiagonal system of equations. $x_1 + 2x_2 = 7$, $x_1 - 3x_2 - x_3 = 4$, $4x_2 + 3x_3 = 5$. Ans: $x_1 = \frac{69}{11}$ $x_2 = \frac{4}{11}$, $x_3 = \frac{13}{11}$.

2 Solve the tridiagonal system of equations. $2x_1 - x_2 = 0$, $x_1 - 2x_2 + x_3 = 0$. $x_2 - 2x_3 + x_4 = 0$, $x_3 - 2x_4 = -1$. Ans:- $x_1 = \frac{1}{5}$ $x_2 = \frac{2}{5}$ $x_3 = \frac{3}{5}$ $x_4 = \frac{4}{5}$.

3 Solve the tridiagonal system of equations $2x - 3y = 8$, $3x + y + z = 4$. $y - 3z = -11$. Ans:- $x = 1$, $y = -2$, $z = 3$.

4 Solve the tridiagonal system $2x_1 - 3x_2 = 5$, $x_1 + 2x_2 - 3x_3 = -1$, $3x_2 - x_3 + 2x_4 = 1$ $x_3 + x_4 = 2$ Ans:- $x_1 = 1$ $x_2 = -1$, $x_3 = 0$, $x_4 = 2$

5 Solve the tridiagonal system $5x + 2y = 3$, $2x - 3y + z = 5$, $4y - 3z = -4$

Ans:- $x = 1$, $y = -1$, $z = 0$.

6 Solve the tridiagonal system $3x_1 + 2x_2 = 1$, $x_1 - 2x_2 + 3x_3 = -2$, $2x_2 - x_3 + x_4 = 1$ and $3x_3 - 4x_4 = 11$ Ans:- $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = -2$

Gaussian Elimination Method :—

This method of solving a system of n linear equations in n unknowns conists of eliminating the co efficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution.

Consider the system of non homogeneous equations.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{——— (1)}$$

The matrix equation of the given system of eqn's is $AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The augmented matrix of this system is.

$$[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$R_2 \longrightarrow R_2 - \frac{a_{21}}{a_{11}} R_1 \quad R_3 \longrightarrow R_3 - \frac{a_{31}}{a_{11}} R_1$, we get

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

where $a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$ $\quad a'_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$

$a'_{32} = a_{32} - \frac{a_{31}}{a_{11}} a_{12}$ $\quad a'_{33} = a_{33} - \frac{a_{31}}{a_{11}} a_{13}$.

$b'_2 = b_2 - \frac{a_{21}}{a_{11}} b_1$ $\quad b'_3 = b_3 - \frac{a_{31}}{a_{11}} b_1$

Here we assume that $a_{11} \neq 0$

We call $-\frac{a_{21}}{a_{11}}$, $-\frac{a_{31}}{a_{11}}$ as multipliers for the first stage.

a_{11} is called first pivot.

$$R_3 \longrightarrow R_3 - \frac{a'_{32}}{a'_{22}} R_2 \text{, we get}$$

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right] \quad \text{——— (2)}$$

$$\text{Where } a''_{33} = a'_{33} - \frac{a'_{32}}{a'_{22}} a'_{23}$$

$$b''_3 = b'_3 - \frac{a'_{32}}{a'_{22}} b'_2$$

We assume that $a'_{22} \neq 0$.

Here the multiplier is $-\frac{a'_{32}}{a'_{22}}$

New pivot is a'_{22}

The augmented matrix (4) corresponds to an upper triangular system which can be solved by backward substitution.

Note :-

(1) If one of the elements a_{11}, a'_{22}, a''_{33} are zero, the method is modified by re arranging the rows so that the pivot is non zero.

(2) This procedure is called partial pivoting.

(3) If this is impossible then the matrix is singular and the system has no solution.

Solve the equations $2x_1+x_2+x_3=10$, $3x_1+2x_2+3x_3=18$, $x_1+4x_2+9x_3=16$ using Gauss Elimination method.

Sol:- Given that

$$\left.\begin{aligned} 2x_1+x_2+x_3&=10\\ 3x_1+2x_2+3x_3&=18\\ x_1+4x_2+9x_3&=16 \end{aligned}\right\} \longrightarrow (1)$$

The matrix equation of the given system of eqn's is $AX=B$

where $A=\begin{bmatrix} 2 & 1 & 1\\ 3 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}$ $X=\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}$ $B=\begin{bmatrix} 10\\ 18\\ 16\end{bmatrix}$

The augmented matrix of the given system is

$$[A|B]=\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10\\ 3 & 2 & 3 & 18\\ 1 & 4 & 9 & 16\end{array}\right]$$

$R_2 \longrightarrow R_2-\frac{3}{2}R_1$ $\quad R_3 \longrightarrow R_3-\frac{1}{2}R_1$

$$\sim\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10\\ 0 & \frac{1}{2} & \frac{3}{2} & 3\\ 0 & \frac{7}{2} & \frac{17}{2} & 11\end{array}\right]$$

$R_3 \longrightarrow R_3-\dfrac{\frac{7}{2}}{\frac{1}{2}}R_2$ $\quad$ i.e $R_3 \longrightarrow R_3-7R_2$

$$\sim\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10\\ 0 & \frac{1}{2} & \frac{3}{2} & 3\\ 0 & 0 & -\frac{4}{2} & -10\end{array}\right] \sim\left[\begin{array}{ccc|c} 2 & 1 & 1 & 10\\ 0 & \frac{1}{2} & \frac{3}{2} & 3\\ 0 & 0 & -2 & -10\end{array}\right]$$

The equivalent matrix equation of the given system of equations is

$$\begin{bmatrix} 2 & 1 & 1\\ 0 & \frac{1}{2} & \frac{3}{2}\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix} 10\\ 3\\ -10\end{bmatrix}$$

The linear equations are

$$2x_1 + x_2 + x_3 = 10$$

$$\frac{x_2}{2} + \frac{3x_3}{2} = 3 \quad \text{i.e} \quad x_2 + 3x_3 = 6$$

$$-2x_3 = -10 \quad \text{i.e} \quad x_3 = 5$$

These equations can be solved by back substitution

$$x_2 = 6 - 3x_3$$

$$x_2 = 6 - 15 = -9$$

$$2x_1 = 10 - x_2 - x_3$$

$$2x_1 = 10 + 9 - 5 = 14$$

$$x_1 = 7.$$

$\therefore$ The solution of the given system is

$$x_1 = 7 \quad x_2 = -9 \quad x_3 = 5.$$

Gauss Jordan Method :—

This is modified Gauss Elimination method.

Consider the given system of linear equations in matrix form $AX = B$

Now reduce the augmented matrix $[A|B]$ by applying E-row operations only such that the coefficient matrix A is in diagonal form $[D|B']$ Then the solution is obtained directly.

(1) Using Gauss Jordan Method, Solve the system

$2x+y+z=10$, $3x+2y+3z=18$, $x+4y+9z=16$.

Sol:- G.T $2x+y+z=10$ $3x+2y+3z=18$ $x+4y+9z=16$.

The matrix equation of the given system of equations is $AX=B$.

where $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$

$R_2 \to 2R_2 - 3R_1$ $R_3 \to 2R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{array}\right]$$

$R_3 \to R_3 - 7R_1$ $R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & -2 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$R_1 \to 2R_1 - R_3$ $R_2 \to 4R_2 + 3R_3$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & 0 & 28 \\ 0 & 4 & 0 & -36 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

This is of the form $[D|B']$.

The equivalent matrix equation of $AX=B$ is.

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ -36 \\ -20 \end{bmatrix}$$

$4x = 28 \Rightarrow x = 7$

$4y = -36 \Rightarrow y = -9$

$-4z = -20 \Rightarrow z = 5$.

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \\ 5 \end{bmatrix}$ is the solution.

(2) Solve the system of equations by Gauss Jordan Method.

(a) $10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$.

Ans:- $x = y = z = 1$.

(b) $10x_1 + x_2 + x_3 = 12$

$x_1 + 10x_2 - x_3 = 10$

$x_1 - 2x_2 + 10x_3 = 9$.

Ans:- $x_1 = x_2 = x_3 = 1$.

GAUSS ELIMINATION METHOD

1 Apply Gauss elimination method solve the equations $x+4y-z=-5$, $x+y-6z=-12$, $3x-y-z=4$.

Ans:- $x=1.6479$, $y=-1.1408$, $z=2.0845$

2 Solve $10x-7y+3z+5u=6$, $-6x+8y-z-4u=5$, $3x+y+4z+11u=2$, $5x-9y-2z+4u=7$ by Gauss elimination method.

Ans:- $x=5, y=4, z=-7, u=1$.

3 solve the following equations by Gauss elimination method.

$2x+y+z=10$, $3x+2y+3z=18$, $x+4y+9z=16$.

Ans:- $x=7$, $y=-9$ $z=5$

4 Solve $2x-y+3z=9$, $x+y+z=6$, $x-y+z=2$ by Gauss elimination method Ans:- $x=2$, $y=2$, $z=3$

5 Solve $2x_1+4x_2+x_3=3$, $3x_1+2x_2-2x_3=-2$, $x_1-x_2+x_3=6$ by Gauss elimination method. Ans: $x_1=2$, $x_2=-1$, $x_3=3$

6 Solve $5x_1+x_2+x_3+x_4=4$, $x_1+7x_2+x_3+x_4=12$, $x_1+x_2+6x_3+x_4=-5$ $x_1+x_2+x_3+4x_4=-6$ Ans:- $x_1=1$, $x_2=2$, $x_3=-1$, $x_4=-2$

7 Solve (if possible) $2x+z=3$, $x-y+z=1$, $4x-2y+3z=3$

Ans:- In consistent.

8 Solve $4x-3y-9z+6w=0$, $2x+3y+3z+6w=6$, $4x-21y-39z-6w=-24$.

Ans:- $x=1+k_1-2k_2$, $y=(4-5k_1-2k_2)/3$ $z=k_1, w=k_2$.

9 Solve $2x_1+x_2+2x_3+x_4=6$, $6x_1-6x_2+6x_3+12x_4=36$, $4x_1+3x_2+3x_3-3x_4=-1$ $2x_1+2x_2-x_3+x_4=10$. Ans:- $x_1=2$, $x_2=1$ $x_3=-1$, $x_4=3$.

10. Solve $2x+3y-z=5$, $4x+4y-3z=3$, $2x-3y+2z=2$.

Ans:- $x=1, y=2, z=3$.

GAUSS JORDAN METHOD

1 Apply Gauss Jordan method, solve the equations $x+y+z=9$, $2x-3y+4z=13$, $3x+4y+5z=40$. Ans:- $x=1$ $y=3$ $z=5$.

2. Solve by Gauss Jordan method $2x+5y+7z=52$, $2x+y-z=0$, $x+y+z=9$

Ans:- $x=1$, $y=3$, $z=5$

3 Solve by Gauss Jordan method $2x-3y+z=-1$, $x+4y+5z=25$, $3x-4y+z=2$ Ans:

4 Solve $x+3y+3z=16$, $x+4y+3z=18$, $x+4z+3y=19$ using Gauss Jordan method. Ans:- $x=1$ $y=2$, $z=3$.

5 Solve $2x+y+z=10$, $3x+2y+3z=18$, $x+4y+9z=16$ using Gauss Jordan method. Ans: $x=7$, $y=-9$, $z=5$

6 Apply Gauss Jordan method solve $2x_1+x_2+5x_3+x_4=5$, $x_1+x_2-3x_3+4x_4=-1$
$3x_1+6x_2-2x_3+x_4=8$, $2x_1+2x_2+2x_3-3x_4=2$

Ans: $x_1=2$ $x_2=\frac{1}{5}$ $x_3=0$ $x_4=\frac{4}{5}$.

7 Solve $5x_1+x_2+x_3+x_4=4$, $x_1+7x_2+x_3+x_4=12$, $x_1+x_2+6x_3+x_4=-5$
$x_1+x_2+x_3+4x_4=-5$ by Gauss Jordan Method.

Ans:- $x_1=1$ $x_2=2$ $x_3=-1$ $x_4=-2$.

8 Solve $2x_1+x_2+2x_3+x_4=6$, $6x_1-6x_2+6x_3+12x_4=36$.
$4x_1+3x_2+3x_3-3x_4=-1$, $2x_1+2x_2-x_3+x_4=10$.

Ans:- $x_1=2$, $x_2=1$, $x_3=-1$, $x_4=3$.

R. NO — Q. NO

Vector :—

An ordered n-tuple of numbers is called an n- vector. The n numbers which are called components of the vector may be written in a horizontal or in a vertical line.

A vector over a real number is called a Real vector and vector over complex numbers is called a complex vector.

Eg.: $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $[1\ 0\ 7]$ are two vectors.

Linearly dependent set of vectors :—

A set $\{x_1, x_2, x_3, \ldots x_r\}$ of r vectors is said to be a linearly dependent set if there exist r scalars $k_1, k_2, k_3 \ldots k_r$ not all zero such that $k_1x_1 + k_2x_2 + k_3x_3 + \cdots + k_rx_r = 0$ where 0 denotes the n vector with components all zero.

Linearly independent set of vectors :—

A set $\{x_1, x_2, x_3, \ldots x_r\}$ of r vectors is said to be linearly independent set if the set is not linearly dependent i.e if $k_1x_1 + k_2x_2 + k_3x_3 + \cdots + k_rx_r = 0$. Where 0 denotes the n vector with components all zero.

(1) show that the system of vectors $(1,3,2)$ $(1,-7,-8)$ $(2,1,-1)$ linearly independent.

sol:- Let $a,b,c \in R$ then

$$a(1,3,2) + b(1,-7,-8) + c(2,1,-1) = \bar{0}$$

$$(a+b+2c,\ 3a-7b+c,\ 2a-8b-c) = (0,0,0)$$

$$a+b+2c=0 \quad 3a-7b+c=0 \quad 2a-8b-c=0.$$

$$a=3 \quad b=1 \quad c=-2.$$

$\therefore$ The given vectors are Linearly dependent.

(2) show that the system of vectors $(1,2,0)$ $(0,3,1)$ $(-1,0,1)$ is Linearly independent.

sol:- Let $a,b,c \in R$ then

$$a(1,2,0) + b(0,3,1) + c(-1,0,1) = \bar{0}$$

$$(a-c,\ 2a+3b,\ b+c) = (0,0,0)$$

$$a-c=0 \quad 2a+3b=0 \quad b+c=0.$$

$$a=0 \quad b=0 \quad b=0.$$

$\therefore$ The given vectors are Linearly independent.

Note:-

(i) If a set of vectors is linearly dependent then atleast one vector of the set can be expressed as a linear combination of the remaining vectors.

(ii) If a set of vectors is linearly independent then no vector of the set can be expressed as a linear combination of the remaining vectors.

System of Homogeneous Linear Equations :—

A set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right] \text{———} (1)$$

is said to be a system of m homogeneous equations in n unknowns

$x_1, x_2, x_3 \ldots x_n$.

The matrix form of the given system of equations (1) is $AX = 0$ ——(2)

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$ is called the coefficient matrix of

the system of equations (1).

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ is the matrix of unknowns and $O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$ is the null

matrix of the system of equations (1).

Note :-

(1) The solution $x_1 = x_2 = x_3 = \cdots = x_n = 0$ i.e $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = 0$ is called

a solution of (1) and it is said to be trivial solution (zero solution)

of the system of homogeneous equations $AX = 0$.

(2) Always the system of homogeneous equations consistent i.e it contains a solution.

(3) Always a system of homogeneous linear equations i.e $AX = 0$ contains $n - r$ linearly independent solutions where r being the

rank of the coefficient matrix A and n being the number of variables of the system.

(4) The trivial solution $x=0$ is not linearly independent and it is a linearly dependent solution.

Nature of solutions of $AX=0$:-

Suppose we have m equations in n unknowns Then the coefficient matrix A will be of order $m \times n$. Let r be the rank of the matrix A.

Case (i) :- If $r=n$, then the given system of equations $AX=0$ will have $n-r=n-n=0$ linearly independent solutions. So in this case the given system possesses a linearly dependent solution i.e only a trivial solution (zero solution).

Case (ii) :- If $r<n$, then the given system of equations $AX=0$ $n-r$ linearly independent solutions. Any linear combination of these solutions will also be a solution of $AX=0$. Thus in this case the given system $AX=0$ contains an infinite number of solutions.

Case (iii) :— Suppose $m<n$ i.e the number of equations less than the number of unknowns. Since $r \leq m$, therefore r is definitely less than n.

Hence in this case the given system of equations must possess a non zero solution. So that the number of solutions of the system $AX=0$ will be infinite.

Working Rule :—

Step 1 :- First write the matrix equation of the given system of equations.

Step 2 :— Reduce the coefficient matrix A to echelon form to determine the rank of A. Let r be the rank of the coefficient matrix A of order $m \times n$ and n be the number of variables or unknowns of the given system of eqn $AX = 0$.

Step 3 :— Case (i) :- If $r = n$, then the given system of equations $AX = 0$ possesses only a trivial trivial solution (zero sol.) i.e $x_1 = 0$ $x_2 = 0$ ... $x_n = 0$ or $X = 0$.

Case (ii) :- If $r < n$, then the given system of equations possesses an infinite number of solutions. Of these solutions, $(n-r)$ solutions are linearly independent and the remaining are depending upon them. So we have to assign arbitrary values for $(n-r)$ variables and the remaining variables are depending upon them.

Case (iii) :- If $m < n$, then since $r \le m < n$, here also the given system possesses an infinite number of solutions.

Note :- (1) If A is a non singular matrix i.e $|A| \neq 0$ then the linear system $AX = 0$ has only a trivial solution (zero solution).

(2) If A is a singular matrix i.e $|A| = 0$, then the linear system $AX = 0$ contains a non zero solution i.e we get an infinite number of solutions.

(1) solve completely the system of equations.

$x+y-3z+2w=0$, $2x-y+2z-3w=0$, $3x-2y+z-4w=0$.

$-4x+y-3z+w=0$.

Sol:- Given that
$$x+y-3z+2w=0$$
$$2x-y+2z-3w=0$$
$$3x-2y+z-4w=0$$
$$-4x+y-3z+w=0$$
$\longrightarrow$ There are 3 eqn's 3 unknowns x, y and z.

The matrix equation of the given system of equations is $AX=0$.

where $A=\begin{bmatrix}1 & 1 & -3 & 2\\ 2 & -1 & 2 & -3\\ 3 & -2 & 1 & -4\\ -4 & 1 & -3 & 1\end{bmatrix}$ is the coefficient matrix of the given system of equations and $X=\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}$

$$A=\begin{bmatrix}1 & 1 & -3 & 2\\ 2 & -1 & 2 & -3\\ 3 & -2 & 1 & -4\\ -4 & 1 & -3 & 1\end{bmatrix}$$

Now we have to reduce the coefficient matrix A to echelon form by applying E-row transformations only and determine the rank of A.

$R_2\longrightarrow R_2-2R_1$, $R_3\longrightarrow R_3-3R_1$ $R_4\longrightarrow R_4+4R_1$

$$\sim\begin{bmatrix}1 & 1 & -3 & 2\\ 0 & -3 & 8 & -7\\ 0 & -5 & 10 & -10\\ 0 & 5 & -15 & 9\end{bmatrix}$$

$R_3\longrightarrow 3R_3-5R_2$, $R_4\longrightarrow 3R_4+5R_2$

$$\sim\begin{bmatrix}1 & 1 & -3 & 2\\ 0 & -3 & 8 & -7\\ 0 & 0 & -10 & 5\\ 0 & 0 & -5 & -8\end{bmatrix}$$

$R_4 \to 2R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & -10 & 5 \\ 0 & 0 & 0 & -21 \end{bmatrix}$$

$\therefore$ $\rho(A) = r = 4 =$ The No. of non zero rows of equivalent to matrix A.

i.e $r = 4 = n$ i.e. the number of unknowns of the given system.

Hence the given system of equations contains only a trivial solution

$\therefore$ $x = y = z = w = 0$ is the only solution of the given system of equations.

→ Solve completely the system of equations

$x - 2y + z - w = 0$ $x + y - 2z + 3w = 0$ $4x + y - 5z + 8w = 0$ and

$5x - 7y + 2z - w = 0$

sol:- Given that $x - 2y + z - w = 0$

$x + y - 2z + 3w = 0$

$4x + y - 5z + 8w = 0$

$5x - 7y + 2z - w = 0$.

→ These are 3 eqns in 4 unknowns x, y and z, w.

The matrix form of the given system of equations is $AX = O$.

Where $A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

Now we reduce the matrix A to echelon form by applying E-row operations only

$R_2 \rightarrow R_2 - R_1 \quad R_3 \rightarrow R_3 - 4R_1 \quad R_4 \rightarrow R_4 - 5R_1$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_2 \,,\; R_4 \rightarrow R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.

Here $\rho(A) = r = 2 =$ The No. of non zero rows equivalent to A.

$\rho(A) = 2 < 4$ (No. of unknowns)

So that the given system possesses an infinite no. of sol's and the remaining of these $n - r = 4 - 2 = 2$ are linearly independent and the remaining are depending upon them.

so we have to assign arbitrary values for 2 variables and the remaining 2 variables are depending upon them.

Now the equivalent matrix eqn of $AX = 0$ is

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn's are

$$x - 2y + z - w = 0$$

$$3y - 3z + 4w = 0.$$

choose $y = K_1 \quad z = K_2$

$$4w = 3z - 3y$$

$$w = \frac{3K_2 - 3K_1}{4}$$

$$x = 2y - z + w$$

$$= 2K_1 - K_2 + \frac{3K_2 - 3K_1}{4}$$

$$x = \frac{5K_1 - K_2}{4}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{5K_1 - K_2}{4} \\ K_1 \\ K_2 \\ \frac{3K_2 - 3K_1}{4} \end{bmatrix} = K_1 \begin{bmatrix} \frac{5}{4} \\ 1 \\ 0 \\ \frac{-3}{4} \end{bmatrix} + K_2 \begin{bmatrix} \frac{-1}{4} \\ 0 \\ 1 \\ \frac{3}{4} \end{bmatrix}$$ is the general

solution of the given system of equations.

→ Solve completely the system of equations $4x + 2y + z + 3u = 0$
$6x + 3y + 4z + 7u = 0$
$2x + y + u = 0$.

Sol:- Given that $4x + 2y + z + 3u = 0$

$$6x + 3y + 4z + 7u = 0$$

$$2x + y + u = 0.$$

→ There are 3 eqn's in 4 unknowns x, y, z and u.

The matrix form of the given system of equations is $AX = O$

where $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Now we reduce the matrix A to echelon form by applying E-row operations only.

$R_2 \longrightarrow 2R_2 - 3R_1 \quad R_3 \longrightarrow 2R_3 - R_1$

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$R_3 \longrightarrow 5R_3 + R_2$

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.

Here $\rho(A) = 2 = r =$ The No. of non zero rows equivalent to A.

$\rho(A) = 2 < 4$ (No. of unknowns)

So that the given system of equations has an infinite no. of solutions. Of these solutions, $n-r = 4-2 = 2$ are linearly independent and the remaining are depending upon them.

So we have to assign arbitrary values for 2 variables and the remaining 2 variables are depending upon them.

Now the equivalent matrix equation of $AX=0$ is

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear equations are

$$4x + 2y + z + 3u = 0$$

$$z + u = 0$$

$$y = k_1$$

$$z = k_2$$

$$u = -z$$

$$u = -k_2$$

$$4x = -2y - z - 3u.$$

$$= -2k_1 - k_2 + 3k_2$$

$$4x = 2k_2 - 2k_1$$

$$x = \frac{k_2 - k_1}{2}$$

$$\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} \frac{k_2 - k_1}{2} \\ k_1 \\ k_2 \\ -k_2 \end{bmatrix} = k_1 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ -1 \end{bmatrix}$$ is the general solution

of given system of equations where k_1, k_2 are arbitrary constants.

Here the two L.I solutions are $X_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

→ Solve the following system of equations for all values of k.

$2x + 3ky + (3k+4)z = 0$, $x + (k+4)y + (4k+2)z = 0$,

$x + 2(k+1)y + (3k+4)z = 0$.

Sol:- Given that the system of equations are

$$\left.\begin{array}{l} 2x + 3ky + (3k+4)z = 0 \\ x + (k+4)y + (4k+2)z = 0 \\ x + 2(k+1)y + (3k+4)z = 0 \end{array}\right] \longrightarrow (1)$$

→ There are 3 eqn's in 3 unknowns x, y and z.

The matrix form of the given system of equations is $AX = 0$ —— (2)

Where $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2(k+1) & 3k+4 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We know that If the coefficient matrix A is singular i.e $|A| = 0$ then the linear system $AX = 0$ contains a non zero solution i.e we get an infinite number of solution.

$|A| = 0$ i.e $\begin{vmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{vmatrix} = 0.$

$R_1 \leftrightarrow R_2$

$$\begin{vmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{vmatrix} = 0$$

$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - R_1$

$$\begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -(k+2) \end{vmatrix} = 0$$

$$(k-2) \begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & 1 & -1 \end{vmatrix} = 0.$$

$$(K-2)\left[\cancel{8}-K+5K\right]=0$$

$$(K-2)(4K+8)=0$$

$$K=\pm 2.$$

Case (i):- When $K \neq \pm 2$, then the given system of equations possesses a zero solution i.e trivial solution. i.e $x=y=z=0$.

Case (ii):- When $K=2$

$$A=\begin{bmatrix} 2 & 6 & 10 \\ 1 & 6 & 10 \\ 1 & 6 & 10 \end{bmatrix}$$

Now reduce the matrix A to echelon form by applying elementary row operations and determine the rank of A.

$$R_2 \rightarrow 2R_2 - R_1, \quad R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 6 & 10 \\ 0 & 6 & 10 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{Which is in echelon form.}$$

$\therefore$ $\rho(A)=2$ = The No. of non zero rows equivalent to A.

$\therefore$ $\rho(A)=2<3$ (No. of unknowns)

So that the given system of eqns Contains an Infinite number of solutions. of these $n-r=3-2=1$ L.I solutions.

We have to assign an arbitrary values for 1 variable. and remaining 2 variables are depending upon them.

The equivalent matrix equation of $AX=0$ is

$$\begin{bmatrix} 2 & 6 & 10 \\ 0 & 6 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear equations are

$$2x + 6y + 10z = 0$$
$$6y + 10z = 0$$

choose $y = K_1$

$$10z = -6y$$
$$z = -\frac{3}{5}K_1$$

$$x = -3y - 5z$$
$$x = -3K_1 - 5.\left(-\frac{3}{5}\right)K_1$$
$$x = 0.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ K_1 \\ -\frac{3}{5}K_1 \end{bmatrix}$$ where K_1 is arbitrary constant.

Case (iii) :- When $K = -2$

$$A = \begin{bmatrix} 2 & -6 & -2 \\ 1 & 2 & -6 \\ 1 & -2 & -2 \end{bmatrix}$$

Now reduce the matrix A to echelon form by applying E-row operations only and determine the rank of A.

$$R_2 \to 2R_2 - R_1, \quad R_3 \to 2R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & -6 & -2 \\ 0 & 10 & -10 \\ 0 & 2 & -2 \end{bmatrix}$$

$$R_3 \to 5R_3 + R_2$$

$$\sim \begin{bmatrix} 2 & -6 & -2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form.

$\therefore \rho(A) = 2 =$ The no. of non zero rows equivalent to A.

$\therefore \; \rho(A) = 2 < 3$ (No. of unknowns)

Sothat the given system of eqn's contains an infinite no. of solutions of these $n-r = 3-2 = 1$ L.I solution.

We have to assign an arbitrary values for $n-r = 3-2 = 1$ variable.

and remaining 2 variables are depending upon them.

The equivalent matrix equation of $AX = 0$ is

$$\begin{bmatrix} 2 & -6 & -2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn's are $2x - 6y - 2z = 0$

$10y - 10z = 0.$

$\Rightarrow y - z = 0.$

choose $y = k_1$

$z = y = k_1.$

$$x = \frac{6y + 2z}{2} = \frac{6k_1 + 2k_1}{2}$$

$x = 4k_1.$

$$\therefore \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$ is the sol. of given system when $k = -2$, where k_1 is an arbitrary constant

→ Find the values of k for which the equations $(k-1)x + (3k+1)y + 2kz = 0$, $(k-1)x + (4k-2)y + (k+3)z = 0$
$2x + (3k+1)y + (3k-3)z = 0$ are consistent and find the ratios of $x:y:z$ when k has the smallest of these values. What happens when k has the greatest of these values.

Sol: Given that $(k-1)x + (3k+1)y + 2kz = 0$

$(k-1)x + (4k-2)y + (k+3)z = 0$

$2x + (3k+1)y + (3k-3)z = 0$

→ These are 3 eqns in 3 unknowns x, y and z

The matrix form of the given system of equations is $AX = 0$.

Where $A = \begin{bmatrix} k-1 & 3k+1 & 2k \\ k-1 & 4k-2 & k+3 \\ 2 & 3k+1 & 3k-3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We know that If the coefficient matrix A is singular i.e $|A| = 0$ then the linear system $AX = 0$ contains a non zero solution i.e we get an infinite number of solutions.

$|A| = 0$ i.e $\begin{vmatrix} k-1 & 3k+1 & 2k \\ k-1 & 4k-2 & k+3 \\ 2 & 3k+1 & 3k-3 \end{vmatrix} = 0$

$R_2 \to R_2 - R_1$

$$\begin{vmatrix} k-1 & 3k+1 & 2k \\ 0 & k-3 & 3-k \\ 2 & 3k+1 & 3k-3 \end{vmatrix} = 0$$

$C_3 \to C_3 + C_2$

$$\begin{vmatrix} k-1 & 3k+1 & 5k+1 \\ 0 & k-3 & 0 \\ 2 & 3k+1 & 6k-2 \end{vmatrix} = 0.$$

$$(k-3)\left[(k-1)(6k+2) - 2(5k+1)\right] = 0.$$

$$(k-3)\,6k\,(k-3) = 0$$

$$k(k-3)^2 = 0$$

$$\therefore\ k = 0, 3, 3.$$

Case (i) When $k = 0$:-

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$$

Now reduce the matrix A into Echelon form by applying elementary row operations only

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is in Echolon form.

$\rho(A) = 2 =$ No. of Non zero rows equivalent to A.

$\rho(A) = 2 < 3$ (No. of unknowns)

So that the given system of equations contains an infinite no. of solutions. Of these $n - r = 3 - 2 = 1$ L.I solution. To determine this, we have to assign an arbitrary values for $n - r = 3 - 2 = 1$ variable.

An equivalent matrix equation of $AX = 0$ is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear equations are

$-x + y = 0$

$-3y + 3z = 0 \Rightarrow y - z = 0.$

Choose $z = k_1$.

$y - z = 0 \implies y = z = k_1$

$-x + y = 0 \implies x = y = k_1$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix}$ is the solution of given system

$\therefore x : y : z = 1 : 1 : 1$

Case (ii) When $k = 3$:-

$$A = \begin{bmatrix} 2 & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{bmatrix}$$

When $k = 3$, The system of equations $AX = 0$ becomes identical.

→ Solve the system completely for all values of λ, $\lambda x + y + z = 0$, $x + \lambda y + z = 0$, $x + y + \lambda z = 0$.

Sol:- Given that $\lambda x + y + z = 0$, $x + \lambda y + z = 0$, $x + y + \lambda z = 0$ —— ①

→ There are 3 eqn's in 3 unknowns x, y and z.

The matrix form of the given system ① is $AX = 0$

Where $A = \begin{bmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

We know that If the coefficient matrix A is singular i.e $|A| = 0$ then the linear system $AX = 0$ contains a non zero solution i.e. we get an infinite no. of solutions.

$$|A| = 0 \quad \text{i.e} \quad \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} \lambda+2 & \lambda+2 & \lambda+2 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$(\lambda+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$C_2 \to C_2 - C_1 \quad C_3 \to C_3 - C_1$$

$$(\lambda+2)\begin{vmatrix} 1 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ 1 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$(\lambda+2)(\lambda-1)^2 = 0$$

$$\lambda = -2, 1, 1.$$

Case(i):- When $\lambda \neq 1, -2$, then the given system of equations posseses a zero solution i.e trivial solution.

$$\therefore x = y = z = 0.$$

Case (ii):- When $\lambda = -2$

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Now reduce the matrix A to echelon form by applying elementary row operations only and determine the rank of A.

$$R_2 \to 2R_2 + R_1, \quad R_3 \to 2R_3 + R_1$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$R_3 \to R_3 + R_2.$$

$$\sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 =$ The No. of non zero rows equivalent to A

$\rho(A) = 2 < 3$ (No. of unknowns)

Sothat the given system of eqn's contains an infinite no.of solutions. of these $n-r = 3-2 = 1$ L.I solution.

To determine this, we have to assign an arbitrary values for $n-r = 3-2 = 1$ variable and remaining 2 variables are depending upon them.

The equivalent matrix eqn. of $AX = 0$ is

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn's are $-2x+y+z=0$

$-3y+3z=0 \Rightarrow y-z=0$

choose $z = K_1$

$y = z = K_1$

$2x = y+z$

$2x = K_1+K_1$

$x = K_1$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} K_1 \\ K_1 \\ K_1 \end{bmatrix}$$ where K_1 is arbitrary constant, is the solution of the given system when $\lambda = -2$.

<u>Case (iii)</u> :- when $\lambda = 1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now reduce the matrix A to echelon form by applying elementary row operations only and determine the rank of A.

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $\rho(A) = 1$ = The No. of non zero rows equivalent to A.

$\rho(A) = 1 < 3$ (No. of unknowns)

Sothat the given system of eqn's contains an infinite no. of solutions.

of these $n-r = 3-1 = 2$ L.I solutions.

To determine this, we have to assign an arbitrary values for $n-r = 3-1 = 2$ variables and remaining 1 variable. is depending upon them.

The equivalent matrix equation of $AX = 0$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn. is $x + y + z = 0$

Choose $y = k_1$, $z = k_2$

$x = -y - z$

$x = -k_1 - k_2$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$ Where k_1 and k_2 are arbitrary constants, is the solution of the given system.

Here the two L.I solutions are $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

→ Show that the only real number λ for which the system.

$x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$, $2x+3y+z=\lambda z$ has a non zero solution is 6 and solve them when $\lambda=6$.

Sol: Given system can be written as
$$\left.\begin{aligned}(1-\lambda)x+2y+3z&=0.\\ 3x+(1-\lambda)y+2z&=0\\ 2x+3y+(1-\lambda)z&=0\end{aligned}\right\} \quad \text{—(1)}$$

These are 3 eqn's in 3 unknowns x, y and z.

The matrix form of the given system is $AX=O$.

Where $A=\begin{bmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix}$ $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ $O=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

We know that If the coeff. matrix A is singular i.e $|A|=0$. then the linear system $AX=0$ contains a non zero solution i.e we get an infinite no. of solutions.

$$|A|=0 \text{ i.e } \begin{vmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$$R_1 \to R_1+R_2+R_3$$

$$\begin{vmatrix}6-\lambda & 6-\lambda & 6-\lambda\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$$(6-\lambda)\begin{vmatrix}1 & 1 & 1\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$$

$$C_2 \to C_2-C_1,\quad C_3 \to C_3-C_1$$

$$(6-\lambda)\begin{vmatrix}1 & 0 & 0\\ 3 & -\lambda-2 & -1\\ 2 & 1 & -\lambda-1\end{vmatrix}=0$$

$$(6-\lambda)\left[(\lambda+2)(\lambda+1)+1\right]=0$$

$$(6-\lambda)(\lambda^2+3\lambda+3)=0$$

$$\lambda=6,\ -\tfrac{3}{2}\pm\tfrac{\sqrt{3}}{2}i$$

$\therefore$ The given system have non zero solution for only real number $\lambda=6$.

Case (i) When $\lambda=6$.

$$A=\begin{bmatrix}-5 & 2 & 3\\ 3 & -5 & 2\\ 2 & 3 & -5\end{bmatrix}$$

Now reduce the matrix A to echelon form by applying elementary row operations only and determine the rank of A.

$R_2 \to 5R_2+3R_1 \quad R_3 \to 2R_1+5R_3$

$$\sim\begin{bmatrix}-5 & 2 & 3\\ 0 & -19 & 19\\ 0 & 19 & -19\end{bmatrix}$$

$R_3 \to R_3+R_2$

$$\sim\begin{bmatrix}-5 & 2 & 3\\ 0 & -19 & 19\\ 0 & 0 & 0\end{bmatrix}$$

Which is in echelon form.

$\rho(A)=2=$ The no. of non zero rows equivalent to A.

$\rho(A)=2<3$ (No. of unknowns).

Sothat the given system of eqn's contains an infinite no. of solutions of these $n-r=3-2=1$ L.I solution.

To determine this we have to assign an arbitrary values for $n-r=3-2=1$ variable and remaining 2 variables are depending upon them.

The equivalent matrix eqn. of $AX = O$ is

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The linear eqn's are $-5x + 2y + 3z = 0$

$$-19y + 19z = 0$$
$$\Rightarrow y - z = 0$$

choose $z = k_1$

$$y = z = k_1$$

$$5x = 2y + 3z$$
$$5x = 2k_1 + 3k_1$$
$$x = k_1$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ where k_1 is an arbitrary constant

is the solution of the given system.

SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS.

1 Find all the solutions of the following homogeneous systems.

(a) $3x+y+2z=0$, $x-2y+3z=0$, $x+5y-4z=0$.

Ans: $x=-k$, $y=z=k$.

(b) $x+y+2z=0$, $3x+4y-7z=0$, $-x-2y+11z=0$.

Ans: $x=-15k$, $y=13k$, $z=k$.

(c) $x+2y+3z+4w=0$, $x+y+z+w=0$, $x+2y+6z+12w=0$.

Ans:- $x=-2\alpha/3$, $y=7\alpha/3$ $z=-8\alpha/3$ $w=\alpha$.

(d) $x+y+z+w=0$, $-x+y+z-w=0$, $-x-y+z+w=0$, $x+y-z+w=0$

Ans:- $x=y=z=w=0$.

(e) $2x-y-3z+w=0$, $x+y+z+w=0$, $2x-7y-13z-w=0$, $-x+5y+9z+w=0$.

Ans: $x=\frac{2}{3}(k_2-k_1)$, $y=-\frac{1}{3}(5k_2+k_1)$ $z=k_2$ $w=k_1$

(f) $3x+y+z+4w=0$, $4y+10z+w=0$, $x+7y+17z+3w=0$, $2x+2y+4z+3w=0$.

Ans:- $x=(2\beta-5\alpha)/4$ $y=-(10\beta+\alpha)/4$ $z=\beta$, $w=\alpha$.

(g) $3x-11y+5z=0$, $4x+y-10z=0$, $4x+9y-6z=0$ Ans:- $x=y=z=0$.

(h) $x+y-3z+2w=0$, $2x-y-2z-3w=0$, $3x-5y-w=0$, $5x-y-7z-4w=0$.

Ans:- $x=(\alpha+5\beta)/3$ $y=(4\beta-7\alpha)/3$ $z=\beta$ $w=\alpha$.

(i) $x+y-2z-w=0$, $2x+y-z-2w=0$, $3x+2y-z-3w=0$, $4x+2y+2z-4w=0$.

Ans:- $x=k_1-2k_2$, $y=5k_2$, $z=k_2$ $w=k_1$.

(j) $3x-11y+5z=0$, $4x+y-10z=0$, $4x+9y-6z=0$ Ans:- $x=y=z=0$.

2 If a, b, c are distinct non zero numbers show that the homogeneous system $\begin{bmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has no non trivial solution.

3 Solve the system $2x+y+2z=0$, $x+y+3z=0$, $4x+3y+8z=0$

Ans:- $x=k$ $y=-4k$ $z=k$.

4 Determine the values of λ for which the following set of equations may possess non trivial solution. $3x_1 + x_2 - \lambda x_3 = 0$, $4x_1 - 2x_2 - 3x_3 = 0$. $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. Ans: $\lambda = 1, -9$; $x_1 = 2k$, $x_2 = 6k$, $x_3 = -\frac{4}{3}k$;

$x_1 = k_1$ $x_2 = -k_1$ $x_3 = 2k_1$.

5 Solve the system of equations $x + 2y + (2+k)z = 0$, $2x + (2+k)y + 4z = 0$. $7x + 13y + (18+k)z = 0$ for all values of k. Ans: $k = 1, \frac{4}{3}$.

$x = 1$ $y = -2k$ $z = k$; $x = \frac{14}{3}k$ $y = -4k$ $z = k$.

6 Solve the system $\lambda x + y + z = 0$, $x + \lambda y + z = 0$, $x + y + \lambda z = 0$ if the system has non zero solution only Ans: $\lambda = 1, -2$, $x = -k_1 - k_2$, $y = k_1$, $z = k_2$.

$x = y = z = k$.

7 Show that the only real number λ for which the system $x + 2y + 3z = \lambda x$, $3x + y + 2z = \lambda y$, $2x + 3y + z = \lambda z$ has non zero solution is 6 and solve them when $\lambda = 6$.

8 Solve $2x + 3ky + (3k+4)z = 0$, $x + (k+4)y + (4k+2)z = 0$, $x + 2(k+1)y + (3k+4)z = 0$

Ans: $k = 2, -2$; $x = 0$ $y = -\frac{5}{3}k_1$ $z = k_1$; $x = 4k_2$, $y = k_2$ $z = k_2$.

9 Find the values of λ for which the equations $(\lambda-1)x + (3\lambda+1)y + 2\lambda z = 0$. $(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$, $2x + (3\lambda+1)y + 3(\lambda-1)z = 0$. are consistent and find the ratio of $x:y:z$ when λ has the smallest of these values. What happens λ has the greatest of these values.

10 Show that the system of equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$ can possess a non trivial solution only if $\lambda = 1$, $\lambda = -3$ obtain the general solution in each case.

11 Solve $4x + 2y + z + 3u = 0$, $2x + y + u = 0$, $6x + 3y + 4z + 7u = 0$

Ans: $x = -\frac{1}{2}(c_1 + c_2)$ $y = c_1$, $z = -c_2$ and $u = c_2$.

12 Solve $x + 3y - 2z = 0$, $2x - y + 4z = 0$, $x - 11y + 14z = 0$.

Ans: $x = -\frac{10}{7}k$, $y = \frac{8}{7}k$ $z = k$.

Problems on L.I and L.D set of vectors :-

→ Examine the following vectors for linear dependence or independence. If dependent, find the relation amongest them.

$x_1 = (2,-1,3,2)$. $x_2 = (3,-5,2,2)$ $x_3 = (1,3,4,2)$

Sol:- Given that $x_1 = (2,-1,3,2)$ $x_2 = (3,-5,2,2)$ $x_3 = (1,3,4,2)$.

Let $k_1 x_1 + k_2 x_2 + k_3 x_3 = \bar{0}$

$k_1(2,-1,3,2) + k_2(3,-5,2,2) + k_3(1,3,4,2) = \bar{0}$.

$(2k_1 + 3k_2 + k_3, -k_1 - 5k_2 + 3k_3, 3k_1 + 2k_2 + 4k_3, 2k_1 + 2k_2 + 2k_3) = (0,0,0,0)$

Equating corresponding components,

$$\begin{aligned} 2k_1 + 3k_2 + k_3 &= 0. \\ -k_1 - 5k_2 + 3k_3 &= 0 \qquad \text{---} \ (1) \\ 3k_1 + 2k_2 + 4k_3 &= 0 \\ 2k_1 + 2k_2 + 2k_3 &= 0. \end{aligned}$$

→ There are 4 eqn's in 3 unknowns.

The matrix form of the system (1) is $AX = O$.

Where $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -5 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$ $X = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -5 & 3 \\ 3 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$$

Now reduce the matrix A into echelon form by applying elementary row operations only.

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -7 & 7 \\ 0 & -5 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 \to 7R_3 - 5R_2, \quad R_4 \to 7R_4 - R_2$

$$\sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Which is in echelon form.

$\rho(A) = 2$ = No. of non zero rows of last equivalent to A.

$\therefore \rho(A) = 2 < 3$ (No. of unknown)

Sothat the given system have an infinite no. of solutions (Non trivial)

$n - r = 3 - 2 = 1$ L.I solution.

Of these $n - r = 3 - 2 = 1$

To determine this we have to assign an arbitrary value for $n-r=3-2=1$

variable and the remaining are depending upon them.

Now the equivalent matrix equation of $Ax = 0$ is.

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$2k_1 + 3k_2 + k_3 = 0$

$-7k_2 + 7k_3 = 0 \Rightarrow k_2 - k_3 = 0.$

choose $k_3 = t$

$k_2 = k_3 = t.$

$k_1 = \dfrac{-3k_2 - k_3}{2} = \dfrac{-3t - t}{2}$

$k_1 = -2t$

$\therefore k_1 = -2t \quad k_2 = t \quad k_3 = t$

Since k_1, k_2, k_3 are not all zero, the vectors are L.D.

We have $k_1 x_1 + k_2 x_2 + k_3 x_3 = 0.$

$-2t x_1 + t x_2 + t x_3 = 0.$

$2x_1 = x_2 + x_3.$

> Examine for linear dependence or independence of vectors $x_1 = (1,1,-1)$ $x_2 = (2,3,5)$ $x_3 = (2,-1,4)$. If dependent find the relation between them.

Sol:- Given that $x_1 = (1,1,-1)$ $x_2 = (2,3,5)$ $x_3 = (2,-1,4)$

Let $k_1 x_1 + k_2 x_2 + k_3 x_3 = \bar{0}$

$$k_1(1,1,-1) + k_2(2,3,5) + k_3(2,-1,4) = \bar{0}$$

$$(k_1 + 2k_2 + 2k_3,\ k_1 + 3k_2 - k_3,\ -k_1 + 5k_2 + 4k_3) = (0,0,0)$$

Equating corresponding components.

$$\left.\begin{aligned} k_1 + 2k_2 + 2k_3 &= 0 \\ k_1 + 3k_2 - k_3 &= 0 \\ -k_1 + 5k_2 + 4k_3 &= 0 \end{aligned}\right\} \quad \text{——— (1)}$$

There are 3 eqn's in 3 unknowns.

The matrix form of the given system (1) is $AX = O$.

where $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -1 \\ -1 & 5 & 4 \end{bmatrix}$ $X = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -1 \\ -1 & 5 & 4 \end{bmatrix}$$

Now reduce the matrix A into echelon form by applying elementary row operations only.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 7 & 6 \end{bmatrix}$$

$R_3 \to R_3 - 7R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 27 \end{bmatrix}$$

→ Which is in echelon form

$\rho(A) = 3 =$ No. of non zero rows of last equivalent to A.

$\therefore \rho(A) = 3 =$ No. of unknown.

Sothat the given system have trivial solution (zero solution)

$$k_1 = 0 \quad k_2 = 0 \quad k_3 = 0.$$

Since k_1, k_2, k_3 are all zero, the vectors are linearly independent.

Linearly independent and Linearly dependent set of vectors.

1) Examine for linear dependence the system of vectors $(1,2,-1,0)$, $(1,3,1,2)$ $(4,2,1,0)$, $(6,1,0,1)$ and if dependent, find the relation between them. Ans:- Linearly independent.

2) Examine whether following vectors are linearly independent or dependent $x_1 = (2,2,1)^T$ $x_2 = (1,3,1)^T$ $x_3 = (1,2,2)^T$

   Ans:- Linearly Independent.

3) Examine whether following vectors are linearly independent or dependent $x_1 = (3,1,1)$ $x_2 = (2,0,-1)$ $x_3 = (4,2,1)$

   Ans:- Linearly independent.

4) Examine whether following vectors are linearly independent or dependent $x_1 = (1,1,-1)$ $x_2 = (2,3,-5)$ $x_3 = (2,-1,4)$

   Ans:- Linearly independent

5) Examine for linear dependence or independence of the following vectors. If dependent, find the relation between them. $x_1 = (1,-1,1)$ $x_2 = (2,1,1)$ $x_3 = (3,0,2)$ Ans:- Linearly dependent, $x_1 + x_2 = x_3$.

6) Examine for linear dependence or independence of the following vectors. If dependent find the relation between them.

   $x_1 = (1,1,1,3)$ $x_2 = (1,2,3,4)$ $x_3 = (2,3,4,7)$

   Ans:- Linearly dependent, $x_1 + x_2 = x_3$.

7) Show that the vectors $x_1 = (1,-1,2,2)^T$ $x_2 = (2,-3,4,-1)^T$, $x_3 = (-1,2,-2,3)^T$ are linearly dependent. Hence find the relation b/n them Ans:- $x_1 = x_2 + x_3$

8) Show that the vectors $x_1 = (3,1,-4)'$ $x_2 = (2,2,-3)$ $x_3 = (0,-4,1)$ are linearly dependent Hence find the relation b/n them.

   Ans:- $2x_1 = 3x_2 + x_3$.

Vector space :-

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every u, v, w in V and scalars c and d, then V is called a vector space (over the reals $\mathbb{R}$)

(1) Addition.

(a) $u+v$ is a vector in V (closure under addition)

(b) $u+v = v+u$. (commutative property of addition)

(c) $(u+v)+w = u+(v+w)$ (Associative property of addition)

(d) There is a zero vector 0 in V suchthat for every u in V. we have $(u+0) = u$ (Additive identity)

(e) For every u in V, there is a vector in V denoted by $-u$ suchthat $u+(-u) = 0$ (Additive inverse)

(2) Scalar multiplication.

(a) cu is in V (closure under scalar multiplication)

(b) $c(u+v) = cu+cv$ (Distributive property of scalar multi.)

(c) $(c+d)u = cu+du$ (Distributive property of scalar multi.)

(d) $c(du) = (cd)u$ (Associative property of scalar multi.)

(e) $1(u) = u$ (Scalar identity property)

Eg :- (1) The set $\mathbb{R}$ of real numbers $\mathbb{R}$ is a vector space over $\mathbb{R}$.

(2) The set $\mathbb{R}^2$ of all ordered pairs of real numbers is a vector space over $\mathbb{R}$.

(3) The set $\mathbb{R}^n$ of all ordered n-tuples of real numers is a vector space over $\mathbb{R}$.

(4) The set $M_{m,n}$ of all $m \times n$ matrices, with real entries is a vector space over $\mathbb{R}$.

(5) The set V of all real valued continuous (differentiable or integrable) functions defined on the closed interval $[a,b]$ is a real vector space with the vector addition and scalar multiplication defined as follows.

$$(f+g)(x) = f(x) + g(x)$$

$$(kf)(x) = k f(x), \quad \text{For all } f, g \in V \text{ and } k \in R.$$

<u>Basis</u> :- If V is any vector space and $S = \{v_1, v_2, v_3 \dots v_n\}$ is a set of vectors in V, then S is called a basis for V if the following two conditions hold.

(i) S is linearly independent

(ii) S spans V.

COMPLEX MATRICES

Conjugate of a Matrix :-

If the elements of matrix A are replaced by their conjugate complexes then the resulting matrix is defined as the conjugate of the given matrix. It is denoted by $\bar{A}$

Eg :- $A = \begin{bmatrix} 7 & 5+4i \\ -2+3i & 4-7i \end{bmatrix}$ $\quad \bar{A} = \begin{bmatrix} 7 & 5-4i \\ -2-3i & 4+7i \end{bmatrix}$

$A = \begin{bmatrix} 0 & 2+3i & 7i \\ 4-7i & 5+3i & 1+i \\ 7 & 1-i & 6+i \end{bmatrix}$ $\quad \bar{A} = \begin{bmatrix} 0 & 2-3i & -7i \\ 4+7i & 5-3i & 1-i \\ 7 & 1+i & 6-i \end{bmatrix}$

Note :- If $\bar{A}$ and $\bar{B}$ be the conjugate matrices of A and B respectively then (i) $\overline{(\bar{A})} = A$

(ii) $\overline{A+B} = \bar{A} + \bar{B}$

(iii) $\overline{kA} = \bar{k}\,\bar{A}$ where k is complex number.

The transpose of the conjugate of square matrix :-

If A is a square matrix and its conjugate is $\bar{A}$, then the transpose of $\bar{A}$ is $(\bar{A})^T$.

It can be easily seen that $(\bar{A})^T = (\overline{A^T})$ i.e the transpose of the conjugate of a square matrix is same as the conjugate of its transpose.

The transposed conjugate of A is denoted by A^θ.

Eg :- $A = \begin{bmatrix} i & 4+3i \\ 3-i & 7 \end{bmatrix}$ $\quad \bar{A} = \begin{bmatrix} -i & 4-3i \\ 3+i & 7 \end{bmatrix}$ $\quad A^\theta = (\bar{A})^T = \begin{bmatrix} -i & 3+i \\ 4-3i & 7 \end{bmatrix}$

Note :- If A^θ and B^θ be the transposed conjugates of A and B respectively then (i) $(A^\theta)^\theta = A$

(ii) $(A \pm B)^\theta = A^\theta \pm B^\theta$

(iii) $(KA)^\theta = \bar{K} A^\theta$ where k is a complex number

(iv) $(AB)^\theta = B^\theta A^\theta$.

Hermitian Matrix

A square matrix A is said to be hermitian if $A^\theta = A$ i.e $(\bar{A})^T = A$

Eg :- $A = \begin{bmatrix} 5 & 2+4i \\ 2-4i & 7 \end{bmatrix}$ $\bar{A} = \begin{bmatrix} 5 & 2-4i \\ 2+4i & 7 \end{bmatrix}$.

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 5 & 2+4i \\ 2-4i & 7 \end{bmatrix} = A$$

$\therefore$ A is hermitian.

Eg :- $A = \begin{bmatrix} 1 & 1+3i & 2-4i \\ 1-3i & 0 & 5-3i \\ 2+4i & 5+3i & 8 \end{bmatrix}$ $\bar{A} = \begin{bmatrix} 1 & 1-3i & 2+4i \\ 1+3i & 0 & 5+3i \\ 2-4i & 5-3i & 8 \end{bmatrix}$

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 1 & 1+3i & 2-4i \\ 1-3i & 0 & 5-3i \\ 2+4i & 5+3i & 8 \end{bmatrix} = A$$

$\therefore$ A is hermitian.

Note :- (i) The elements of the principal diagonal of a hermitian matrix must be real.

(ii) A Hermitian matrix over the field of real numbers is nothing but a real symmetric matrix.

Skew Hermitian Matrix :-

A square matrix A is said to be skew hermitian if $A^{\theta} = -A$

Eg :- $A = \begin{bmatrix} 0 & 2-3i \\ -2-3i & i \end{bmatrix}$ $\quad \bar{A} = \begin{bmatrix} 0 & 2+3i \\ -2+3i & -i \end{bmatrix}$

$$A^{\theta} = (\bar{A})^T = \begin{bmatrix} 0 & -2+3i \\ 2+3i & -i \end{bmatrix}$$

$$A^{\theta} = -\begin{bmatrix} 0 & 2-3i \\ -2-3i & +i \end{bmatrix}$$

$$A^{\theta} = -A$$

$\therefore$ A is skew hermitian

Eg :- $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ $\quad \bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$

$$A^{\theta} = (\bar{A})^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

$$A^{\theta} = -A$$

$\therefore$ A is skew hermitian.

Note :- (i) The elements of the principal diagonal of a skew hermitian matrix must be purely imaginary or zero.

(ii) A skew hermitian matrix over the field of real numbers is nothing but a real skew symmetric matrix.

Unitary Matrix :-

A square matrix A is said to be unitary if $AA^{\theta} = A^{\theta}A = I$.

Eg:- $A = \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$ $\quad \bar{A} = \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$

$$A^{\theta} = (\bar{A})^{T} = \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$$

$$AA^{\theta} = \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}\frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+3 & \sqrt{3}i - \sqrt{3}i \\ -\sqrt{3}i+\sqrt{3}i & 3+1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^{\theta} = I$$

$\therefore$ A is unitary matrix .

Note :- A unitary matrix over the field of real numbers is nothing but a real orthogonal matrix .

Properties :-

(i) If A is hermitian then iA is skew hermitian .

(ii) If A is skew hermitian then iA is hermitian .

(iii) The matrix $B^{\theta}AB$ is hermitian or skew hermitian according as A is hermitian or skew hermitian .

(iv) The transpose of unitary matrix is unitary .

(v) The inverse of unitary matrix is unitary .

(vi) The product of two unitary matrices is unitary .

(vii) The determinant of unitary matrix is of unit modulus .

Properties of Complex matrices :-

Theorem :- If A is a Hermitian then iA is skew Hermitian.

proof :- Let A be a Hermitian matrix so that $A^{\theta} = A$

$$\text{Now} \quad (iA)^{\theta} = \bar{i} A^{\theta} \qquad [\because (KA)^{\theta} = \bar{K} A^{\theta}]$$

$$= (-i) A^{\theta} \qquad [\because A^{\theta} = A]$$

$$= -iA$$

$$(iA)^{\theta} = -iA$$

$\Rightarrow iA$ is a skew Hermitian matrix.

Eg:- If $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ then prove that A is Hermitian and iA is skew Hermitian.

Sol:- Given $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$$

$$A^{\theta} = \bar{A}^{T} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} = A$$

$\therefore$ A is Hermitian.

$$iA - i \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4i & i+3 \\ i-3 & 7i \end{bmatrix}$$

$$\overline{iA} = \begin{bmatrix} -4i & -i+3 \\ -i-3 & -7i \end{bmatrix}$$

$$(\overline{iA})^{T} = \begin{bmatrix} -4i & -3-i \\ 3-i & -7i \end{bmatrix} = -\begin{bmatrix} 4i & 3+i \\ i-3 & 7i \end{bmatrix}$$

$$(\overline{iA})^{T} = (iA)^{\theta} = -iA$$

$\therefore$ iA is skew Hermitian

Theorem :- If A is a skew Hermitian then iA is Hermitian.

Proof :- Let A be a skew Hermitian matrix so that $A^\theta = -A$

$$(iA)^\theta = \bar{i}\, A^\theta$$
$$= (-i)(-A)$$
$$= iA$$
$$(iA)^\theta = iA$$

iA is Hermitian matrix.

Eg: If $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ then prove that A is skew Hermitian matrix and iA is Hermitian matrix.

Sol:- Given that $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix}$$

$$A^\theta = \bar{A}^T = -\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} = -A$$

$\therefore A^\theta = -A$

$\therefore A$ is skew Hermitian.

$$iA = i\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} = \begin{bmatrix} -3 & 2i-1 \\ -2i-1 & 1 \end{bmatrix}$$

$$\overline{iA} = \begin{bmatrix} -3 & -2i-1 \\ 2i-1 & 1 \end{bmatrix}$$

$$(\overline{iA})^T = \begin{bmatrix} -3 & 2i-1 \\ -2i-1 & 1 \end{bmatrix}$$

$$(iA)^\theta = (\overline{iA})^T = iA$$

$$(iA)^\theta = iA$$

$\therefore iA$ is ~~skew~~ Hermitian.

Theorem :- The matrix $B^\theta AB$ is Hermitian or skew Hermitian according as A is Hermitian or skew Hermitian.

Proof :- (i) Let A be a Hermitian matrix so that $A^\theta = A$.

Now $(B^\theta AB)^\theta = B^\theta A^\theta (B^\theta)^\theta$

$= B^\theta A^\theta B$. $[\because (B^\theta)^\theta = B]$

$= B^\theta A B$

$(B^\theta AB)^\theta = B^\theta AB$.

$\Rightarrow B^\theta AB$ is a Hermitian matrix.

(ii) Let A be a skew Hermitian matrix so that $A^\theta = -A$.

$(B^\theta AB)^\theta = B^\theta A^\theta (B^\theta)^\theta$

$= B^\theta A^\theta B$

$= B^\theta (-A) B$

$= -B^\theta AB$

$\therefore (B^\theta AB)^\theta = -B^\theta AB$.

$\Rightarrow B^\theta AB$ is skew Hermitian matrix.

Theorem :- The transpose of unitary matrix is unitary.

Proof :- Let A be the unitary matrix so that $AA^\theta = I = A^\theta A$

Now $(AA^\theta)^T = I^T = (A^\theta A)^T$ (Taking Transpose)

$(A^\theta)^T A^T = I = A^T (A^\theta)^T$

$(A^T)^\theta A^T = I$

$\Rightarrow A^T$ is unitary matrix.

Eg :- Prove that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is unitary matrix and A^T is also unitary matrix.

Sol: Given that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$\bar{A} = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$

$A^\theta = (\bar{A})^T = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

$$AA^{\theta} = \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is unitary.

$$A^T = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}$$

$$\overline{A^T} = \frac{1}{2}\begin{bmatrix} 1-i & +1-i \\ -1-i & 1+i \end{bmatrix}$$

$$(A^T)^{\theta} = (\overline{A^T})^T = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ +1-i & 1+i \end{bmatrix}$$

$$(A^T)^{\theta} A^T = \frac{1}{4}\begin{bmatrix} 1-i & -1-i \\ +1-i & 1+i \end{bmatrix}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A^T)^{\theta} A^T = I$$

$\therefore$ A^T is unitary.

Theorem :- The inverse of unitary matrix is unitary.

Proof :- Let A be unitary matrix so that $AA^{\theta} = I = A^{\theta}A$

Now $(AA^{\theta})^{-1} = I^{-1} = (A^{\theta}A)^{-1}$ (Taking inverse)

$$(A^{\theta})^{-1}(A^{-1}) = I = (A^{-1})(A^{\theta})^{-1}$$

$$(A^{-1})^{\theta} A^{-1} = I = A^{-1}(A^{-1})^{\theta}$$

$\therefore$ A^{-1} is unitary matrix.

Eg :- If $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ then prove that A and A^{-1} are unitary matrices.

Sol: Given that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad A^{\theta} = (\bar{A})^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$AA^{\theta} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore \; AA^{\theta} = I$

A is unitary.

$$|A| = \frac{-1}{2} - \frac{1}{2} = -1.$$

$$A^{-1} = \frac{1}{|A|} \text{Adj} A$$

$$A^{-1} = -1 \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\overline{A^{-1}} = - \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(\overline{A^{-1}})^{T} = (A^{-1})^{\theta} = - \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$(A^{-1})^{\theta} A^{-1} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \; (A^{-1})^{\theta} A^{-1} = I.$

$\therefore \; A^{-1}$ is unitary.

Theorem:- The product of two unitary matrices is unitary.

Proof:- Let A and B be two unitary matrices.

$\Rightarrow AA^{\theta} = I = A^{\theta}A$ and $BB^{\theta} = I = B^{\theta}B$

We prove that AB is unitary.

Consider $(AB)^{\theta}(AB) = (B^{\theta}A^{\theta})(AB)$

$= B^{\theta}(A^{\theta}A)B$

$= B^{\theta} I B$

$= B^{\theta}B = I$

$(AB)^{\theta}(AB) = I$

$\Rightarrow$ AB is unitary

Hence if A and B are unitary then AB is also unitary.

Similarly $(AB)(AB)^{\theta} = (AB)(B^{\theta}A^{\theta})$

$$= A(BB^{\theta})A^{\theta}$$

$$= AIA^{\theta}$$

$$= AA^{\theta} = I.$$

$\therefore (AB)(AB)^{\theta} = (AB)^{\theta}(AB) = I.$

$\therefore$ AB is a unitary matrix.

Eg If $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ and $B = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ are unitary then p/T AB is unitary

Sol: Given that $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ $B = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$AB = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$AB = \frac{1}{\sqrt{6}}\begin{bmatrix} 1+i+1 & +1+i-i \\ -i-1+i & -i+1+1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2+i & 1 \\ -1 & 2-i \end{bmatrix}$$

$$\overline{AB} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2-i & 1 \\ -1 & 2+i \end{bmatrix}$$

$$(AB)^{\theta} = (\overline{AB})^{T} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2-i & -1 \\ 1 & 2+i \end{bmatrix}$$

$$(AB)^{\theta}(AB) = \frac{1}{6}\begin{bmatrix} 2-i & -1 \\ 1 & 2+i \end{bmatrix}\begin{bmatrix} 2+i & 1 \\ -1 & 2-i \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(AB)^{\theta}(AB) = I$$

$\therefore$ AB is unitary

Theorem :- The determinant of a unitary matrix is of unit modulus.

Proof :- Let A be unitary so that $AA^{\theta} = I$.

$$\Rightarrow |AA^{\theta}| = |I| \qquad \because |AB| = |A||B|$$

$$\Rightarrow |A||A^{\theta}| = 1$$

$$\Rightarrow |A|\,|(\bar{A})^{T}| = 1$$

$$\Rightarrow |A|\,|\bar{A}| = 1 \qquad [\because |B| = |B^{T}|]$$

$$\Rightarrow |A|^{2} = 1$$

$\Rightarrow |A|$ is of unit modulus.

Hence if A is unitary then $|A|$ is of unit modulus.

Eg:- If $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$, then prove that $|A|$ is of unit modulus.

Sol:- Given that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$$|A| = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1$$

$$|A| = -1$$

$\therefore$ A is unitary and its determinant is of unit modulus.

Eg:- Prove that the determinant of $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is of unit modulus.

Sol: Given that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$$|A| = \begin{vmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{vmatrix}$$

$$= \frac{2}{4} - \left(-\frac{2}{4}\right)$$

$$|A| = 1$$

$\therefore$ A is unitary and its determinant is of unit modulus.

Theorem :- Every square matrix is uniquely expressed as the sum of Hermitian and skew Hermitian matrices.

Proof :- Let A be a square matrix

Consider $A = \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A-A^{\theta})$

$A = P + Q$ where $P = \frac{1}{2}(A+A^{\theta})$ $Q = \frac{1}{2}(A-A^{\theta})$.

We prove that P is Hermitian and Q is skew Hermitian matrices.

$$P = \frac{1}{2}(A+A^{\theta})$$

$$P^{\theta} = \left[\frac{1}{2}(A+A^{\theta})\right]^{\theta} = \frac{1}{2}(A+A^{\theta})^{\theta}$$

$$= \frac{1}{2}\left[A^{\theta} + (A^{\theta})^{\theta}\right]$$

$$= \frac{1}{2}(A^{\theta}+A)$$

$$P^{\theta} = P.$$

$\therefore$ P is Hermitian matrix.

$$Q = \frac{1}{2}(A-A^{\theta})$$

$$Q^{\theta} = \left[\frac{1}{2}(A-A^{\theta})\right]^{\theta}$$

$$= \frac{1}{2}(A-A^{\theta})^{\theta}$$

$$= \frac{1}{2}(A^{\theta}-(A^{\theta})^{\theta})$$

$$= \frac{1}{2}(A^{\theta}-A)$$

$$= -\frac{1}{2}(A-A^{\theta})$$

$$Q^{\theta} = -Q$$

$\therefore$ Q is skew Hermitian matrix

Thus every square matrix can be expressed as the sum of Hermitian and skew Hermitian matrices.

Uniqueness :-

Let $A = R+S$ be another such representation of A, where R is Hermitian and Q is skew Hermitian.

Then we have to p.t $P=R$ and $Q=S$.

$$P = \frac{1}{2}(A+A^{\theta})$$

$$= \frac{1}{2}\left[(R+S) + (R+S)^{\theta}\right]$$

$$= \frac{1}{2}\left[(R+S) + (R^{\theta}+S^{\theta})\right]$$

$$= \frac{1}{2}\left[R+S+R-S\right] = \frac{1}{2}(2R)$$

$$P = R.$$

$$Q = \frac{1}{2}(A-A^{\theta})$$

$$= \frac{1}{2}\left[(R+S) - (R+S)^{\theta}\right]$$

$$= \frac{1}{2}\left[(R+S) - (R^{\theta}+S^{\theta})\right]$$

$$= \frac{1}{2}\left[(R+S) - (R-S)\right]$$

$$= \frac{1}{2}(2S)$$

$$Q = S$$

$\therefore$ $P=R$ and $Q=S$

Hence the representation is unique.

(1) Express the matrix $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ as the sum of a Hermitian and a skew Hermitian matrices.

Sol:- Given that $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -i & 2+3i & 4-5i \\ 6-i & 0 & 4+5i \\ i & 2+i & 2-i \end{bmatrix}$$

$$A^{\theta} = (\bar{A})^{T} = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

Hermitian part of the matrix A is $P = \frac{1}{2}(A + A^{\theta})$

$$P = \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}\left\{\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} + \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}\right\}$$

$$P = \frac{1}{2}\begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$$

This is a Hermitian matrix

$$Q = \frac{1}{2}(A - A^{\theta}) = \frac{1}{2}\left\{\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} - \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}\right\}$$

$$Q = \frac{1}{2}\begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

This is a skew Hermitian matrix.

$$P + Q = \frac{1}{2}\begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2i & 4-6i & 8+10i \\ 12+2i & 0 & 8-10i \\ -2i & 4-2i & 4+2i \end{bmatrix}$$

$$= \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} = A$$

$\therefore$ $A = P + Q$ where P is Hermitian and Q is skew Hermitian.

Prove that every Hermitian matrix can be written as $P+iQ$ where P is a real symmetric matrix and Q is a real skew symmetric matrix. 29

Sol:- Let A be a Hermitian matrix.

$$A^{\theta} = A$$

$$A = \frac{1}{2}(A+\bar{A}) + i\,\frac{1}{2i}(A-\bar{A}) = P+iQ.$$

Where $P = \frac{1}{2}(A+\bar{A})$ and $Q = \frac{1}{2i}(A-\bar{A})$ are real matrices.

$$P^{T} = \left[\frac{1}{2}(A+\bar{A})\right]^{T} = \frac{1}{2}\left[A^{\theta}+\bar{A}\right]^{T}$$

$$= \frac{1}{2}\left[(\bar{A})^{T}+\bar{A}\right]^{T} = \frac{1}{2}\left[\left[(\bar{A})^{T}\right]^{T}+(\bar{A})^{T}\right]$$

$$= \frac{1}{2}\left(\bar{A}+A^{\theta}\right) = \frac{1}{2}(\bar{A}+A)$$

$$P^{T} = P.$$

Hence P is a real symmetric matrix.

Also, $$Q^{T} = \left[\frac{1}{2i}(A-\bar{A})\right]^{T} = \frac{1}{2i}\left[A^{\theta}-\bar{A}\right]^{T}$$

$$= \frac{1}{2i}\left[(\bar{A})^{T}-\bar{A}\right]^{T} = \frac{1}{2i}\left[\left[(\bar{A})^{T}\right]^{T}-(\bar{A})^{T}\right]$$

$$= \frac{1}{2i}\left[\bar{A}-A^{\theta}\right] = \frac{1}{2i}(\bar{A}-A)$$

$$= -\frac{1}{2i}(A-\bar{A}) = -Q$$

$$Q^{T} = -Q.$$

Hence Q is a real skew symmetric matrix.

Thus, every Hermitian matrix can be written as $P+iQ$, where P is a real symmetric matrix and Q is a real skew symmetric matrix.

Express the Hermitian matrix $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$ as $P + iQ$ where P is a real symmetric matrix and Q is a real skew symmetric matrix.

sol: Given that $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}$$

$$A = \frac{1}{2}(A+\bar{A}) + i\,\frac{1}{2i}(A-\bar{A}) = P + iQ.$$

where $P = \frac{1}{2}(A+\bar{A})$, $Q = \frac{1}{2i}(A-\bar{A})$

Let $P = \frac{1}{2}(A+\bar{A}) = \frac{1}{2}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} + \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$

$$P = \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$Q = \frac{1}{2i}(A-\bar{A}) = \frac{1}{2i}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2+3i \\ 1-i & 2+3i & 2 \end{bmatrix} - \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$

$$Q = \frac{1}{2i}\begin{bmatrix} 0 & -2i & 2i \\ 2i & 0 & -6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$P^T = P, \quad Q^T = -Q$$

We know that P is a real symmetric matrix and Q is a real skew. Symmetric matrix.

$$A = P + iQ = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} + i\begin{bmatrix} 0 & -i & i \\ i & 0 & -3i \\ -i & 3i & 0 \end{bmatrix}$$

Show that every square matrix can be uniquely expressed as $P+iQ$ where P and Q are Hermitian matrices. 26

Sol: Let A be a square matrix.

$$A = \frac{1}{2}(A+A^\theta) + i\frac{1}{2i}(A-A^\theta) = P+iQ.$$

where $P = \frac{1}{2}(A+A^\theta)$ and $Q = \frac{1}{2i}(A-A^\theta)$

Now $P^\theta = \frac{1}{2}(A+A^\theta)^\theta = \frac{1}{2}\left[A^\theta + (A^\theta)^\theta\right]$

$$= \frac{1}{2}(A^\theta + A) = P$$

$$P^\theta = P$$

Hence, P is a Hermitian matrix.

$$Q^\theta = \left[\frac{1}{2i}(A-A^\theta)\right]^\theta = -\frac{1}{2i}\left[A^\theta - (A^\theta)^\theta\right]$$

$$= -\frac{1}{2i}\left[A^\theta - A\right]$$

$$= \frac{1}{2i}\left[A - A^\theta\right]$$

$$Q^\theta = Q$$

Hence Q is a Hermitian matrix.

Thus, every square matrix can be expressed as $P+iQ$ where P and Q are Hermitian matrices

Uniqueness:- Let $A = R+iS$ where R and S are Hermitian matrices.

$$A^\theta = (R+iS)^\theta = R^\theta + (iS)^\theta = R - iS \qquad R^\theta = R$$

$$S^\theta = S.$$

$$\frac{1}{2}(A+A^\theta) = \frac{1}{2}\left[(R+iS)+(R-iS)\right] = R = P$$

$$\frac{1}{2}(A-A^\theta) = \frac{1}{2}\left[(R+iS)-(R-iS)\right] = iS = iQ.$$

Hence, representation $A = P+iQ$ is unique.

Express the matrix $A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$ as $P + iQ$ where P and Q are both Hermitian.

sol:

$$A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -2i & -3 & 1+i \\ 0 & 2-3i & 1-i \\ 3i & 3-2i & 2+5i \end{bmatrix} \qquad A^{\theta} = \bar{A}^{T} = \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}$$

Let $P = \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix} + \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix}$$

$Q = \frac{1}{2i}(A - A^{\theta}) = \frac{1}{2i}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix} - \begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}$

$$= \frac{1}{2i}\begin{bmatrix} 4i & -3 & 1-4i \\ 3 & 6i & -2+3i \\ -1-4i & 2+3i & -10i \end{bmatrix}$$

We know that P and Q are Hermitian matrices.

$$A = P + iQ = \frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4i & -3 & 1-4i \\ 3 & 6i & -2+3i \\ -1-4i & 2+3i & -10i \end{bmatrix}$$

Prove that every skew Hermitian matrix can be written as $P+iQ$ where P is a real skew symmetric matrix and Q is a real symmetric matrix.

sol: Let A be a skew Hermitian matrix.

$$A^{\theta} = -A$$

$$A = \frac{1}{2}(A+\bar{A}) + i\,\frac{1}{2i}(A-\bar{A}) = P+iQ$$

Where $P = \frac{1}{2}(A+\bar{A})$ and $Q = \frac{1}{2i}(A-\bar{A})$ are real matrices.

$$P^T = \left[\frac{1}{2}(A+\bar{A})\right]^T = \frac{1}{2}\left[-A^{\theta}+\bar{A}\right]^T$$

$$= \frac{1}{2}\left[-(\bar{A})^T + \bar{A}\right]^T$$

$$= \frac{1}{2}\left[-\left[(\bar{A})^T\right]^T + (\bar{A})^T\right]$$

$$= \frac{1}{2}\left[-\bar{A} + A^{\theta}\right]$$

$$= \frac{1}{2}\left[-\bar{A} - A\right] = -\frac{1}{2}(A+\bar{A}) = -P$$

$$P^T = -P$$

Hence P is a real skew symmetric matrix.

$$Q^T = \left[\frac{1}{2i}(A-\bar{A})\right]^T = \frac{1}{2i}\left(-A^{\theta} - \bar{A}\right)^T$$

$$= \frac{1}{2i}\left[-(\bar{A})^T - \bar{A}\right]^T = \frac{1}{2i}\left[-\left[(\bar{A})^T\right]^T - (\bar{A})^T\right]$$

$$= \frac{1}{2i}\left[-\bar{A} - A^{\theta}\right] = \frac{1}{2i}\left[-\bar{A}+A\right] = \frac{1}{2i}(A-\bar{A}) = Q$$

$$Q^T = Q.$$

Hence Q is a real symmetric matrix.

Thus, every skew Hermitian matrix can be written as $P+iQ$ where P is a real skew symmetric matrix and Q is a real symmetric matrix.

Express the skew Hermitian matrix $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $P+iQ$ where P is a real skew symmetric matrix and Q is a real symmetric matrix.

Sol:

$$A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}$$

Let $P = \frac{1}{2}(A+\bar{A}) = \frac{1}{2}\left\{\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1+i & 3i & 0 \end{bmatrix} + \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}\right\}$

$$P = \frac{1}{2}\begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$Q = \frac{1}{2i}(A-\bar{A}) = \frac{1}{2i}\left\{\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1+i & 3i & 0 \end{bmatrix} - \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}\right\}$$

$$= \frac{1}{2i}\begin{bmatrix} 4i & 2i & -2i \\ 2i & -2i & 6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$P^T = -P, \quad Q^T = Q.$$

We know that P is a real skew symmetric matrix and Q is a real symmetric matrix.

$$A = P + iQ = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2i & i & -i \\ i & -i & 3i \\ -i & 3i & 0 \end{bmatrix}.$$

COMPLEX MATRICES

1 Define complex matrix. Give an example.

2 Define Conjugate of a matrix. Give an example.

3 Define Conjugate transpose of a matrix Give an example.

4 Define Hermitian matrix Gives an example.

5 Define skew hermitial matrix Give an example.

6 Define Unitary matrix. Give an example.

7 (a) If A is Hermitian matrix then prove that iA is skew Hermitian. matrix

(b) If $A=\begin{bmatrix}-1 & 2+i & 5-3i\\ 2-i & 7 & 5i\\ 5+3i & 5i & 2\end{bmatrix}$ show that A is a Hermitian matrix and $B=iA$ is a skew Hermitian matrix.

8 (a) If A is skew Hermitian matrix then prove that iA is Hermitian matrix.

(b) If $A=\begin{bmatrix}-i & 3+2i & -2-i\\ -3+2i & 0 & 3-4i\\ 2-i & -3-4i & -2i\end{bmatrix}$ show that A is skew Hermitian matrix and $B=iA$ is a Hermitian matrix.

9 Express the matrix $A=\begin{bmatrix}1+i & -i & 2-3i\\ 2 & 1+2i & 3+i\\ -1+i & 3 & 1-2i\end{bmatrix}$ as the sum of a Hermitian matrix and a skew Hermitian matrix

Ans:- $P=\frac{1}{2}\begin{bmatrix}2 & 2-i & 1-4i\\ 2+i & 2 & 6+i\\ 1+4i & 6-i & 2\end{bmatrix}$ $Q=\frac{1}{2}\begin{bmatrix}2i & -i-2 & 3-2i\\ 2-i & 4i & i\\ -3-2i & i & -4i\end{bmatrix}$

10 (a) Show that the matrix $A=\begin{bmatrix}i & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{bmatrix}$ is both a skew Hermitian matrix and a unitary matrix.

(b) Verify that the matrix $A=\frac{1}{2}\begin{bmatrix}1+i & -1+i\\ 1+i & 1-i\end{bmatrix}$ is a unitary matrix.

11 (a) Show that the matrix $A=\begin{bmatrix}a+ic & -b+id\\ b+id & a-ic\end{bmatrix}$ is unitary if $a^2+b^2+c^2+d^2=1$.

(b) If $A=\begin{bmatrix}2+i & 3 & -1+3i\\ -5 & i & 4-2i\end{bmatrix}$ Show that AA^* is a Hermitian matrix.

12 If $A=\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ show that $B=(I-A)(I+A)^{-1}$ is a unitary matrix

13 Find the Eigen values and Eigen vectors of the matrix $A=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

Ans :- $\lambda = 9, 2$ $\quad X_1 = \begin{bmatrix} -1+3i \\ 2 \end{bmatrix}$ $\quad X_2 = \begin{bmatrix} 1-3i \\ 5 \end{bmatrix}$

14 Find the Eigen values and Eigen vectors of the matrix $A=\begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix}$

Ans:- $\lambda = 1+\sqrt{10}\,i,\ 1-\sqrt{10}\,i$

15 Find the Eigen values and Eigen vectors of the matrix $A=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

Ans :- $\lambda = 1, -1$ $\quad x_1 = \begin{bmatrix} 1 \\ i-i\sqrt{2} \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ i+i\sqrt{2} \end{bmatrix}$

MODULE -II

EIGEN VALUES AND EIGEN VECTORS

EIGEN VALUES AND EIGEN VECTORS.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$ be a square matrix. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ be a column vector. Consider the equation $AX = \lambda X$ — ①. Where λ is a scalar. If I is a unit matrix of order n then the equation ① can be written as $AX = \lambda I X$.

$$AX - \lambda I X = 0.$$

$$(A - \lambda I) X = 0 \quad — ②.$$

This matrix equation represents the following system of n homogeneous equations in n unknowns.

$$\left.\begin{aligned} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n &= 0. \\ a_{31}x_1 + a_{32}x_2 + (a_{33}-\lambda)x_3 + \cdots + a_{3n}x_n &= 0. \\ - - - - - - - - - - - - \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n &= 0 \end{aligned}\right\} — ③$$

Here the coefficient matrix of this system is $A - \lambda I$.

We know that the necessary and sufficient condition for the system ③ posseses a non zero solution is that the coefficient matrix $A - \lambda I$ is singular i.e $|A - \lambda I| = 0$.

<u>Characteristic Matrix</u> :- Let A be a square matrix of order n and I be a unit matrix of order n. Then the matrix $A - \lambda I$ is called characteristic matrix where λ is a constant.

Characteristic Polynomial :—

The determinant of the matrix $A-\lambda I$ is called characteristic polynomial in λ of degree n.

Characteristic Equation :—

For a square matrix A, the equation $|A-\lambda I|=0$ is called the characteristic equation.

Eigen values :- The roots of the characteristic equation are called the characteristic values or roots or Eigen values or Latent roots or proper values of the square matrix.

Note :- The set of the Eigen values of A is called the spectrum of A.

Eigen vectors :- If λ is an eigen value of the square matrix A then $\det(A-\lambda I)=0$ i.e The matrix $A-\lambda I$ is singular. Therefore there exists a non zero vector X such that $(A-\lambda I)X=0$ or $AX=\lambda X$ is said to be the eigen vector or characteristic vector of A corresponding to the eigen ~~values~~. values.

(OR)

Let A be a square matrix of order n. A Non zero vector X is said to be characteristic vector of A if there exists a scalar λ such that $AX=\lambda X$.

Note :- An Eigen value of a square matrix A can be zero. But a zero vector can not be an Eigen vector of A.

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Properties of Eigen values and Eigen vectors :—

1) The sum of the Eigen values of a square matrix is equal to its trace of the matrix.

i.e If $\lambda_1, \lambda_2, \lambda_3$ are Eigen values of A then $tr(A) = \lambda_1 + \lambda_2 + \lambda_3$.

Eg :- (i) If 2, 3, 5 are Eigen values of A then $tr(A) = 2+3+5 = 10$

(ii) If 0, 1, −1 are Eigen values of A then $tr(A) = 0+1-1 = 0$.

2) The product of the Eigen values of a square matrix is equal to its determinant.

i.e If $\lambda_1, \lambda_2\ \lambda_3$ are Eigen values of A then $|A| = \lambda_1 \lambda_2 \lambda_3$.

Eg :- (i) If 0, 0, 1 are Eigen values of A then $|A| = 0.0.1 = 0$.

(ii) If 1, 3, −5 are Eigen values of A then $|A| = 1.3.(-5) = -15$.

Note :- (i) If one of the Eigen values of A is zero then A is singu-lar matrix

(ii) If all the Eigen values of A are non zero then A is non singu-lar matrix.

3) If λ is an eigen value of A corresponding to the eigen vector X. then λ^n is an eigen value of A^n corresponding to the eigen vector X.

Eg :- If −1, 1, 2 are Eigen values of A then Eigen values of A^3 are $(-1)^3$, 1^3 and 2^3 i.e −1, 1, 8.

4) If λ is an eigen value of A corresponding to the eigen vector X. then $K\lambda$ is an eigen value of KA corresponding to the eigen vector X. Where K is non zero scalar.

Eg :- If 1, 2, 3 are Eigen values of A then Eigen values of 3A are 3, 6, and 9.

5) If λ is an eigen value of a non singular matrix A correspo

- nding to the eigen vecto X, then λ^{-1} is an eigen value of A^{-1}.

corresponding to the eigen vector X.

Eg:- If 1, 2, 3 are eigen values of A then eigen values of

A^{-1} are 1^{-1}, 2^{-1} and 3^{-1} i.e 1, $\frac{1}{2}$ and $\frac{1}{3}$.

6) If λ is an eigen value of a non singular matrix A correspon

- ding to the eigen vector X then $\frac{|A|}{\lambda}$ is an eigen value of the

matrix adjA corresponding to the eigen vector X.

Eg:- If 1, 3, 5 are eigen values of A then eigen values of

adj A are given by $\frac{|A|}{\lambda} = \frac{15}{1}, \frac{15}{3}, \frac{15}{3}$ i.e 15, 3, 5

$[\because |A| = 1.3.5 = 15]$.

7) If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also

an eigen value.

8) The Eigen values of a triangular matrix are just the diagonal

element of the matrix.

Eg:- An Eigen values of $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$ are $\lambda = 1, -3$.

9) For a real symmetric matrix, the eigen vectors corresponding

to two distinct eigen values are orthogonal.

Eg:- $X_1 = [-16 \ 1 \ 11]^T$ $X_2 = [2 \ -1 \ 3]^T$ $X_3 = [1 \ 5 \ 1]^T$ are eigen.

vectors of corresponding to distinct eigen values of real

symmetric matrix. Here X_1, X_2 and X_3 are pairwise orthogonal.

10) If X is an Eigen vector of a matrix A, then X can not correspond to more than one eigen value of A.

11) The Eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.

12) If X_1 and X_2 are two Eigen vectors of a matrix A corresponding to some some eigen value λ then any linear combination $K_1X_1 + K_2X_2$ where K_1, K_2 are arbitrary constants is also an eigen vector of A corresponding to the same Eigen value λ.

13) A square matrix A and its transpose A^T have the same eigen values.

Eg:- If 2, 3 are eigen values of A then eigen values A^T are 2,3.

14) If λ is an eigen value of the matrix A then $\lambda + K$ is an eigen value of the matrix $A + KI$ corresponding to the eigen vector X.

Eg: (i) If 1, 2, 3 are eigen values of A then eigen values of $A + 2I$ are $1+2, 2+2, 3+2$ i.e 3, 4 and 5.

(ii) If 0, 1, −2 are eigen values of A then eigen values of $A - 3I$ are $0-3, 1-3, -2-3$ i.e −3, −2 and −5.

15) An Eigen values of skew hermitian matrix are purely imaginary or zero.

16) An Eigen values of hermitian matrix are real.

17) The Eigen values of an unitary matrix have absolute value 1.

Working procedure to find Eigen values :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

step(i) :- The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$.

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \text{ ——— } (1)$$

Where S_1 = Sum of the principal diagonal elements of A i.e $tr(A)$

$S_1 = a_{11} + a_{22} + a_{33}$.

S_2 = Sum of the minors of principal diagonal elements of A

$$S_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$S_3 = \det A$.

step(ii) :- Find S_1, S_2 and S_3,

step (iii) :- substitute the values of S_1, S_2 and S_3 in (1)

solve the eqn. (1), we get Eigen values λ_1, λ_2 and λ_3.

If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(a) verify that $|A| = \lambda_1 \lambda_2 \lambda_3$ and $tr(A) = \lambda_1 + \lambda_2 + \lambda_3$.

(b) Find the eigen values for the following matrices

(i) A (ii) A^T (iii) A^{-1} (iv) $4A^{-1}$ (v) A^2 (vi) A^2-2A+I (vii) A^3+2I.

(viii) $A-2I$.

Sol:- Given that $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$

i.e $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0.$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

Where S_1 = sum of the principal diagonal elements of $A = 3+5+3 = 11$

S_2 = sum of the minors of principal diagonal elements of A

$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$

$= (15-1) + (9-1)(15-1)$

$S_2 = 36$

$S_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(15-1) + 1(-3+1) + 1(1-5)$

$S_3 = 36.$

$\therefore$ The characteristic equation of A is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\lambda = 2, 3, 6.$

(a) $|A| = 2.3.6 = 36$, $tr(A) = 2+3+6 = 11.$

(b) (i) Eigen values of $A = \lambda$ $\longrightarrow 2, 3, 6$

(ii) Eigen values of $A^T = \lambda$ $\longrightarrow 2, 3, 6$

(iii) Eigen values of $A^{-1} = \lambda^{-1}$ $\longrightarrow \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

(iv) Eigen values of $4A^{-1} = 4\lambda^{-1}$ $\longrightarrow \frac{4}{2}, \frac{4}{3}, \frac{4}{6}$

(v) Eigen values of $A^2 = \lambda^2$ $\longrightarrow 2^2, 3^2, 6^2$

(vi) Eigen values of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$ $\longrightarrow 1, 4, 25$

(vii) Eigen values of $A^3 + 2I = \lambda^3 + 2$ $\longrightarrow 10, 29, 218$

(viii) Eigen values of $A - 2I = \lambda - 2$ $\longrightarrow 0, 1, 4.$

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Working procedure to find Eigen values and Eigen vectors :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

step(i) :- The characteristic equation of A is $|A-\lambda I| = 0$.

i.e $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$.

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad \text{——} \ (1)$$

Where $S_1 = tr(A)$

S_2 = sum of the minors of principal diagonal elements of A

$S_3 = |A|$.

step (ii) :- solve the characteristic eqn. (1), we get Eigen values λ_1, λ_2 and λ_3.

step (iii) For finding an Eigen vector corresponding Eigen value $\lambda = \lambda_1$ we solve homogeneous system $(A-\lambda I)X = 0$. —— (2).

i.e $\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Similarly we can find an eigen vector corresponding eigen value $\lambda = \lambda_2$, $\lambda = \lambda_3$. by solving homogeneous system (2).

→ Determine the characteristic roots and the characteristic vectors of the matrix $A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$. Also find characteristic roots and char. vectors of (i) A^2 (ii) A^{-1}.

Sol:- Given that $A=\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$

Let λ be the eigen value of A.

The characteristic equation of A is $|A-\lambda I|=0$.

$$\text{i.e} \begin{vmatrix}2-\lambda&1&0\\0&2-\lambda&1\\0&0&2-\lambda\end{vmatrix}=0$$

$$(2-\lambda)^3=0$$

$$\lambda=2,2,2. \quad \left[\text{Algebraic multiplicity of } \lambda=2 \text{ is } 3\right]$$

∴ Eigen values of A are $\lambda=2,2,2$.

Now the Eigen vector $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ of A corresponding to the Eigen value λ are obtained by solving the homogeneous system of eqns

$(A-\lambda I)X=0$ i.e $\begin{bmatrix}2-\lambda&1&0\\0&2-\lambda&1\\0&0&2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ —— ①

Eigen vector corresponding to Eigen value $\lambda=2$:-

For $\lambda=2$, The system ① can be written as

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Which is in echelon form.

Here rank of the coefficient matrix of the system is 2 i.e $r=2$

So that the system has $n-r=3-2=1$ L.I solution.

There is only one L.I eigen vector corresponding to Eigen value $\lambda=2$.

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To determine this, We have to assign an arbitrary value for $n-r = 3-2 = 1$ variable.

From the above system, the eqn's can be written as.

$$x_2 = 0 , \quad x_3 = 0 .$$

Note that we can not find x_1 from these eqn's. As x_1 is not present in any of these equations, it follows that x_1 can be arbitrary

$$\text{Hence } x_1 = k_1 \quad x_2 = 0, \; x_3 = 0$$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}$ is the only linearly independent Eigen vector of A Corresponding to the Eigen value $\lambda = 2$. (Geometric multiplicity of $\lambda = 2$ is 1).

(i) We know that If λ is an eigen value of A corresponding to the Eigen vector x then λ^n is an eigen value of A^n corresponding to the Eigen vector X.

$\therefore$ Eigen values of A^2 is $\lambda^2 = 2^2, 2^2, 2^2$. and the corresponding eigen vector is $X_1 = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}$.

(ii) We know that If λ is an eigen value of A corresponding the Eigen vector x then λ^{-1} is an eigen value of A^{-1} corresponding to the Eigen vector X.

$\therefore$ Eigen values of A^{-1} is $\lambda^{-1} = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and the corresponding Eigen vector is $X_1 = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}$.

→ Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$.
Also find eigen values and eigen vectors of
(i) adj A (ii) $A - 3I$.

Sol:- Given that $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e.} \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(-2-\lambda) = 0$$

$$\lambda = 1, 2, -2$$

∴ Eigen values of the matrix A are $\lambda = 1, 2, -2$

[Algebraic multiplicity of $\lambda = 1, 2, -2$ is 1]

Now the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to Eigen value λ are obtained by solving the homogeneous system $(A - \lambda I)X = 0$.

$$\text{i.e.} \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—} (1)$$

(i) Eigen vector corresponding to the Eigen value $\lambda = 1$:-

For $\lambda = 1$, The system (1) can be written as

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by applying E-row operations only. and hence determine the rank of coeff. matrix.

$$R_2 \to 2R_2 - R_1$$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to 5R_3 + 3R_2$$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form

Here the rank of the coeff. matrix of the system is 2 i.e $r=2$

Sothat the system has $n-r = 3-2 = 1$ L.I solution.

There is only one L.I eigen vector corresponding to eigen value $\lambda=1$.

To determine this we have to assign an arbitrary value for $n-r=$ $3-2=1$ variable.

From the above system, the eqn's can be written as

$$2x_2 - x_3 = 0$$

$$5x_3 = 0 \Rightarrow x_3 = 0$$

$$x_2 = \frac{x_3}{2}$$

$$x_2 = 0$$

Now we can't find x_1 from these equations. As x_1 is not present in any of these eqn's. it follows that x_1 is an arbitrary.

Hence $x_1 = k_1,\ x_2 = 0,\ x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the linearly independent eigen vector corresponding to eigen value $\lambda = 1$.

[Geometric multiplicity of $\lambda=1$ is 1]

Case (ii) Eigen vector corresponding to Eigen value $\lambda = 2$:-

For $\lambda=2$, The system (1) can be written as

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the co eff. matrix into echelon form by applying E-row operations only and hence determine the rank of Coeff. matrix

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the Coeff. matrix of the system is 2 i.e $r=2$

Sothat the system has $n-r = 3-2 = 1$ L.I solution.

There is only one L.I eigen vector corresponding to the eigen value $\lambda=2$.

To determine this we have to assign an arbitrary value for $n-r = 3-2 = 1$ variable.

From the above system, the eqn's can be written as

$$-x_1 + 2x_2 - x_3 = 0$$

$$2x_3 = 0 \Rightarrow x_3 = 0$$

$$-x_1 + 2x_2 = 0$$

$$\Rightarrow x_1 = 2x_2$$

choose $x_2 = k_2$ Then $x_1 = 2k_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_2 \\ k_2 \\ 0 \end{bmatrix} = k_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

$\therefore X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ is the L.I eigen vector corresponding eigen value $\lambda = 2$.

[Geometric multiplicity of $\lambda=2$ is 1]

Case (iii) Eigen vector corresponding to the Eigen value $\lambda=-2$:-

For $\lambda=-2$, The system ① can be written as

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the coeff. matrix of the system is 2 i.e $r=2$

So that the system has $n-r = 3-2 = 1$ L.I solution.

There is only one L.I eigen vector corresponding to eigen value $\lambda=-2$.

To determine we have to assign an arbitrary value for $n-r =$ $3-2=1$ variable.

From the above system, the equations can be written as

$$3x_1 + 2x_2 - x_3 = 0$$

$$4x_2 + 2x_3 = 0 \implies x_2 = -\frac{1}{2}x_3$$

choose $x_3 = k_3$

$$x_2 = -\frac{1}{2}k_3$$

$$x_1 = \frac{x_3 - 2x_2}{3}$$

$$x_1 = \frac{2}{3}k_3.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}k_3 \\ -\frac{1}{2}k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \text{ where } k_3 \neq 0.$$

$\therefore X_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda=-2$

[Geometric multiplicity of $\lambda=-2$ is 1]

$\therefore$ The Eigen values of A are 1, 2, −2 and the corresponding eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2/3 \\ -1/2 \\ 1 \end{bmatrix}$.

(i) We know that λ is an eigen value of non singular matrix A. Corresponding to the eigen vector x then $\frac{|A|}{\lambda}$ is an eigen value of adj A corresponding to the eigen vector X.

$\therefore$ Eigen values of adj A are $\frac{|A|}{\lambda} = \frac{-4}{1}, \frac{4}{2}, \frac{4}{-2}$ i.e $-4, 2, -2$.

and corresponding eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} \frac{2}{3} \\ -1/2 \\ 1 \end{bmatrix}$

(ii) We know that λ is an eigen value of A corresponding to the eigen vector x then $\lambda - k$ is an eigen value of $A - kI$ corresponding to the eigen vector X.

$\therefore$ Eigen values of $A - 3I$ are $\lambda - 3 = -2, -1, -5$ and corresponding eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2/3 \\ -1/2 \\ 1 \end{bmatrix}$.

Find the eigen values and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Sol:- Given that $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$.

$$\text{i.e} \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 - R_2$

$$\begin{vmatrix} -\lambda & \lambda & 0 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$\begin{vmatrix} -\lambda & \lambda & 0 \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix} = 0.$$

$$\lambda^2 \begin{vmatrix} -1 & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -1 \end{vmatrix} = 0$$

$C_2 \to C_2 + C_3$

$$\lambda^2 \begin{vmatrix} -1 & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & -1 \end{vmatrix} = 0.$$

$$\lambda^2 \left[[-1(\lambda - 2) - 0] - (-1) \right] = 0.$$

$$\lambda^2 [2 - \lambda + 1] = 0 \Rightarrow \lambda^2 (3 - \lambda) = 0$$

$$\lambda = 0, 0, 3.$$

The Eigen values of the matrix A are $\lambda = 0, 0, 3$.

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Now the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to Eigen value λ

are obtained by solving the homogeneous system $(A-\lambda I)X = 0$.

i.e $$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ --- } (1)$$

Eigen vector corresponding to the Eigen value $\lambda = 0$:—

For $\lambda = 0$, The system (1) can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and hence determine the rank of the coefficient matrix.

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the Rank of the coefficient matrix of the system is 1 i.e $r = 1$.

So that the system has $n - r = 3 - 1 = 2$ linearly independent sol's.

There are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 0$.

To determine this, from the above system. the eqns can be written as $\quad x_1 + x_2 + x_3 = 0$

choose $x_2 = k_1$
$x_3 = k_2$

$$x_1 = -x_2 - x_3$$

$$x_1 = -k_1 - k_2.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are the L.I eigen vectors corrosponding to the eigen value $\lambda = 0$.

Eigen vector corrosponding to the eigen value $\lambda = 3$:—

For $\lambda = 3$, The system (1) can be written as.

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and. hence determine the rank of the coefficient matrix.

$$R_2 \to 2R_2 + R_1, R_3 \to 2R_3 + R_1$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the Rank of the coefficient matrix of the system is 2 i.e $r=2$ so that the system has $n-r = 3-2 = 1$ L.I sol.

There is only one L.I eigen vector corrosponding to the eigen value $\lambda = 3$.

To determine this, from the above system, the eqn's can be written as $-2x_1 + x_2 + x_3 = 0$

$$x_2 - x_3 = 0$$

choose $x_3 = k_1$

$$x_2 = x_3$$

$$x_2 = k_1$$

$$2x_1 = x_2 + x_3$$

$$x_1 = k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{where } k_1 \neq 0$$

$X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the L.I eigen vector corresponding to the eigen value. $\lambda = 3$.

$\therefore$ The Eigen value of A are 0, 0, 3 and the corresponding to the eigen vectors are $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

② show that $A=\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is a skew Hermitian matrix and also 19

Find eigen values and the corrosponding eigen vectors of A.

sol:- Given that $A=\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

$$\bar{A}=\begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$A^{\theta}=\bar{A}^{T}=\begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$A^{\theta}=-\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}-A$$

$\therefore$ A is skew hermitian matrix.

The characteristic equation of A is $|A-\lambda I|=0$

i.e $$\begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{vmatrix}=0$$

$$(i-\lambda)(\lambda^2+1)=0$$

$$\lambda=-i, i, i$$

The eigen values of the matrix A are $\lambda=-i, i, i$

Now we have to find eigen vectors $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corrosponding to the eigen values of λ by solving the homogeneous system $(A-\lambda I)X=0$.

i.e $$\begin{bmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——①.}$$

Case(i):- Eigen vector corrosponding to the eigen value $\lambda = -i$

For $\lambda = -i$, The system ① can be written as

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and determine the rank of the matrix.

$R_3 \longrightarrow R_3 - R_2$

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the rank of the coefficient matrix of the system is $r = 2$ = The No. of non zero rows.

So that the system have $n - r = 3 - 2 = 1$ linearly independent sol.

$\therefore$ There is only one linearly independent eigen vector corrosponding to the eigen value $\lambda = -i$

To determine this, we have to assign an arbitrary value for $n - r = 3 - 2 = 1$ variable.

The linear equations are $x_1 = 0$

$x_2 + x_3 = 0$

choose $x_3 = k_1$

$x_2 = -x_3 = -k_1$.

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$x_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corrosponding to the eigen value $\lambda = -i$.

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case (iii) :- Eigen vector corrosponding to the eigen value $\lambda = i$:-

For $\lambda = i$, The system (1) can be written as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now we reduce the co efficient matrix to echelon form by applying elementary row operations only and determine the rank of the matrix.

$$R_3 \longrightarrow R_3 + R_2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the rank of the co efficient matrix of the system is $r = 1 =$ The No. of non zero rows.

so that the system have $n - r = 3 - 1 = 2$ linearly independent solutions. There are only two linearly independent eigen vectors corrosponding to the eigen value $\lambda = i$

To determine this, we have to assign an arbitrary value for $n - r = 3 - 1 = 2$ variables.

The linear equation is $x_2 - x_3 = 0$.

choose $x_3 = k_2$

$x_2 = x_3 = k_2$.

Now we can not find x_1 from these equations. As x_1 is not present in any of these equations it follows that x_1 is an aeblitrary.

Hence $x_1 = k_3$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_3 \\ k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent eigen vectors corresponding to the eigen value $\lambda = i$

$\therefore$ $X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are eigen vectors corresponding to the eigen values $\lambda = -i, i, i$.

Find the eigen values and eigen vectors of the Hermitian matrix.

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$$A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$$

Sol: Given that $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$ i.e $\begin{vmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{vmatrix} = 0$.

$$(2-\lambda)^2 - (3+4i)(3-4i) = 0.$$

$$(2-\lambda)^2 - 25 = 0$$

$$\lambda^2 - 4\lambda - 21 = 0.$$

$$\lambda^2 - 7\lambda + 3\lambda - 21 = 0$$

$$\lambda(\lambda-7) + 3(\lambda-7) = 0$$

$$(\lambda-7)(\lambda+3) = 0$$

$$\lambda = -3, 7.$$

The eigen values of the matrix A are $\lambda = -3, 7$.

Now we have find the eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponding to the eigen values of λ by solving the homogeneous system $(A-\lambda I)X = 0$.

$$\text{i.e } \begin{bmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \text{—(1).}$$

Case (i) :- Eigen vector corresponding to the eigen value $\lambda = 7$.

For $\lambda = 7$ the system (1) can be written as

$$\begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now we reduce the coefficient matrix to echelon form by applying E-row operations only and determine the rank of the matrix

$$R_2 \rightarrow 5R_2 + (3-4i)R_1$$

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$$\begin{bmatrix} -5 & 3+4i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here the rank of the coefficient matrix of the system is $r=1$.

So that the system have $n-r = 2-1 = 1$ linearly independent solutions.

$\therefore$ There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = 7$.

To determine this, we have to assign an arbitrary value for $n-r = 2-1 = 1$ variable.

The linear eqn is, $-5x_1 + (3+4i)x_2 = 0$.

choose $x_2 = k_1$

$$x_1 = \frac{3+4i}{5} x_2$$

$$x_1 = \frac{3+4i}{5} k_1$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3+4i}{5} k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$X_1 = \begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$.

case(ii):- Eigen vector corresponding to the eigen value $\lambda = -3$:-

For $\lambda = -3$, The system ① can be written as.

$$\begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

. Now we reduce the co efficient matrix to echelon form by applying elementary row operations only and determine the rank of the matrix.

$$R_2 \rightarrow 5R_2 - (3-4i)R_1$$

$$\begin{bmatrix} 5 & 3+4i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here the rank of the co efficient matrix of the system is $r=1$

so that the system have $n-r = 2-1 = 1$ linearly independent solution.

There is only one linearly independent eigen vector corrosponding to the eigen value $\lambda = -3$

To determine this, we have to assign an arbitrary value for $n-r = 2-1 = 1$ variable.

The linear eqn is $5x_1 + (3+4i)x_2 = 0$

choose $x_2 = k_2$

$$5x_1 = -(3+4i)x_2$$

$$x_1 = -\frac{(3+4i)}{5} k_2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{(3+4i)}{5} k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} -\frac{(3+4i)}{5} \\ 1 \end{bmatrix}$$ where $k_2 \neq 0$.

$X_2 = \begin{bmatrix} -\frac{(3+4i)}{5} \\ 1 \end{bmatrix}$ is the L.I eigen vector corrosponding to the eigen value $\lambda = -3$.

$\therefore X_1 = \begin{bmatrix} \frac{3+4i}{5} \\ 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} -\frac{(3+4i)}{5} \\ 1 \end{bmatrix}$ are the eigen vectors corrosponding to the eigen values $\lambda = 7, -3$.

Find the eigen values and corrosponding eigen vectors of the matrix 21

$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$.

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sol:- Given that $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e} \begin{vmatrix} \frac{1}{\sqrt{3}} - \lambda & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} - \lambda \end{vmatrix} = 0.$$

$$-\left(\frac{1}{3} - \lambda^2\right) - \frac{(1+i)(1-i)}{\sqrt{3}\sqrt{3}} = 0$$

$$\lambda^2 - \frac{1}{3} - \frac{1}{3}(1+1) = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1.$$

The eigen values of the matrix A are $\lambda = 1, -1$.

Now we have to find eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corrosponding to eigen values of λ by solving the homogeneous system $(A-\lambda I)X = 0$

$$\text{i.e} \begin{bmatrix} \frac{1}{\sqrt{3}} - \lambda & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (1).$$

Case(i):- Eigen vector corresponding to the eigen value $\lambda = 1$

For $\lambda = 1$, The system ① can be written as.

$$\begin{bmatrix} \frac{1}{\sqrt{3}}-1 & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}}-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to \left(\frac{1-\sqrt{3}}{\sqrt{3}}\right)R_2 - \left(\frac{1-i}{\sqrt{3}}\right)R_1.$$

$$\begin{bmatrix} \frac{1-\sqrt{3}}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here the rank of the coefficient matrix of the system is

$r = 1 =$ No. of non zero rows .

So that the system have $n-r = 2-1=1$ linearly independent solutions

$\therefore$ There is only one linearly indepent eigen vector corresponding to the eigen value $\lambda=1$.

To determine this, we have to assign an arbitrary value for $n-r = 2-1 = 1$ variable.

The linear equation is $\left(\frac{1-\sqrt{3}}{\sqrt{3}}\right)x_1 + \left(\frac{1+i}{\sqrt{3}}\right)x_2 = 0$

choose $x_2 = k_1$

$$\frac{1-\sqrt{3}}{\sqrt{3}}x_1 = \frac{-(1+i)}{\sqrt{3}}x_2$$

$$x_1 = -\frac{(1+i)}{1-\sqrt{3}}k_1$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1+i}{\sqrt{3}-1}k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} \frac{1+i}{\sqrt{3}-1} \\ 1 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$X_1 = \begin{bmatrix} \frac{1+i}{\sqrt{3}-1} \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$

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case (ii):- Eigen vector corresponding to the eigen value $\lambda = -1$

For $\lambda = -1$, The system (1) can be written as.

$$\begin{bmatrix} \frac{1}{\sqrt{3}}+1 & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{-1}{\sqrt{3}}+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1+\sqrt{3}}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & \frac{-1+\sqrt{3}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and determine the rank of the matrix.

$$R_2 \longrightarrow \left(\frac{1+\sqrt{3}}{\sqrt{3}}\right) R_2 - \left(\frac{1-i}{\sqrt{3}}\right) R_1$$

$$\begin{bmatrix} \frac{1+\sqrt{3}}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Here the rank of the coefficient matrix of the system is $r = 1 =$ No. of non zero rows.

So that the system have $n - r = 2 - 1 = 1$ linearly independent solution.

$\therefore$ There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = -1$.

To determine this, we have to assign an arbitrary value for

$n-r = 2-1 = 1$ variable

The linear equation is

$$\left(\frac{1+\sqrt{3}}{\sqrt{3}}\right)x_1 + \left(\frac{1+i}{\sqrt{3}}\right)x_2 = 0$$

choose $x_2 = k_2$

$$(1+\sqrt{3})\,x_1 = -(1+i)\,x_2$$

$$x_1 = \frac{-(1+i)}{1+\sqrt{3}}\,k_2$$

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{(1+i)}{1+\sqrt{3}}k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} -\frac{(1+i)}{1+\sqrt{3}} \\ 1 \end{bmatrix} \text{ where } k_2 \neq 0.$$

$x_2 = \begin{bmatrix} -\frac{(1+i)}{1+\sqrt{3}} \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = -1$.

$\therefore\ x_1 = \begin{bmatrix} \frac{1+i}{\sqrt{3}-1} \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -\frac{(1+i)}{\sqrt{3}+1} \\ 1 \end{bmatrix}$ are two linearly independent eigen vectors corresponding to the eigen values $\lambda = 1, -1$.

Determine the constants p, q, r, s, t, u so that $[1\ 1\ 1]^T$, $[1, 0\ -1]^T$ and $[1\ -1\ 0]^T$ are the eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the eigen vector corrosponding to λ_1.

$$\therefore \quad AX_1 = \lambda_1 X_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{bmatrix}$$

$1+1+1 = \lambda_1$ i.e $\lambda_1 = 3$.

$p+q+r = \lambda_1 \implies p+q+r = 3$. —— ①

$s+t+u = \lambda_1 \implies s+t+u = 3$ —— ②

Let $x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ be the eigen vector corrosponding to λ_2 Then

$$AX_2 = \lambda_2 X_2$$

$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ 0 \\ -\lambda_2 \end{bmatrix}$$

$1 \cdot 1 + 1 \cdot 0 + 1(-1) = \lambda_2 \implies \lambda_2 = 0$.

$p + q \cdot 0 + r(-1) = 0 \implies p - r = 0$. —— ③

$s + t \cdot 0 - u = -\lambda_2 \implies s - u = 0$ —— ④.

Let $x_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ be the eigen vector corrosponding to λ_3 Then

$$AX_3 = \lambda_3 X_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ s & t & u \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \lambda_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_3 \\ -\lambda_3 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 0.$$

$$p - q = -\lambda_3 \Rightarrow p - q = 0. \text{ --- } (5)$$

$$s - t = 0 \text{ --- } (6).$$

To get the values of p, q, r, s, t, u we have to solve the equations (1) to (6).

$$(1) + (3) + (5) \Rightarrow 3p = 3 \Rightarrow p = 1.$$

$\therefore$ (3) $\Rightarrow r = 1$ and (5) $\Rightarrow q = 1$.

Similarly from (2), (4) and (6) we get $s = t = u = 1$.

$$\therefore p = q = r = s = t = u = 1.$$

$\therefore$ The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Let a 3×3 matrix A have eigen values $1, 2, -1$. Find the trace of the matrix $B = A - A^{-1} + A^2$. Also find determinant of B.

Sol:- Given that the eigen values of the matrix A are $\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = -1$.

We know that If λ is an eigen value of the matrix A then. $f(\lambda)$ is an eigen value of the matrix $f(A)$.

$$\text{Let } f(A) = A - A^{-1} + A^2.$$

An eigen values of the matrix A^2 are $\lambda_1 = 1$ $\lambda_2 = 4$ $\lambda_3 = 1$

An eigen values of the matrix A^{-1} are $\lambda_1 = 1$ $\lambda_2 = \frac{1}{2}$ $\lambda_3 = -1$.

$$\text{Let } f(\lambda) = \lambda - \lambda^{-1} + \lambda^2.$$

$$\therefore f(\lambda_1) = f(1) = 1 - 1 + 1 = 1.$$

$$f(\lambda_2) = f(2) = 2 - \frac{1}{2} + 4 = \frac{11}{2}.$$

$$f(\lambda_3) = f(-1) = -1 - (-1) + 1 = 1.$$

$\therefore$ The eigen values of the matrix $f(A)$ i.e B are $1, \frac{11}{2}$ and 1.

$\therefore$ The trace of the matrix $B = 1 + \frac{11}{2} + 1 = \frac{15}{2}$.

The determinant of the matrix $B = 1 \cdot \frac{11}{2} \cdot 1 = \frac{11}{2}$.

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EIGEN VALUES AND EIGEN VECTORS

1) Find the Eigen values and Eigen vectors of a matrix A and A^3.

Where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

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Ans:- Eigen values of A are $\lambda = -2, 3, 6$; Eigen values of A^3 are $\lambda = -8, 27, 196$

Eigen vectors $x_1 = [1, 0, -1]^T$ $x_2 = [1, -1, 1]^T$ $x_3 = [1, 2, 1]^T$

2) Determine the Eigen values and Eigen vectors of A and A^{-1}

Where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Ans:- Eigen values of A are $\lambda = 1, 2, 3$; Eigen values of A^{-1} are $\lambda = 1, \frac{1}{2}, \frac{1}{3}$.

Eigen vectors $x_1 = [-1, 1, 0]^T$ $x_2 = [-2, 1, 2]^T$ $x_3 = [-1, 1, 2]^T$.

3) Determine the Eigen values and Eigen vectors of A and Adj A.

Where $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Ans:- Eigen values of A are $\lambda = 1, 2, -2$; Eigen values of Adj A are $\lambda = -4, -2, 2$.

Eigen vectors $x_1 = [-1, 1, 1]^T$ $x_2 = [0, 1, 1]^T$ $x_3 = [8, -5, 3]^T$.

4) Find the Eigen values and Eigen vectors a matrix A and 2A, 30A, 44A.

Where $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Ans:- Eigen values of (i) A are $\lambda = 1, 2, -2$ (ii) 2A are 2, 4, −4

(iii) 30A are 30, 60, −60 (iv) 44, 88, −88.

Eigen vectors are $x_1 = [1, 0, 0]^T$ $x_2 = [2, 1, 0]^T$ $x_3 = [-\frac{4}{3}, 1, -2]^T$

5 If $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ find the eigen values and eigen vectors of A and those of

$B = 2A^2 - \frac{1}{2}A + 3I$. Ans:- Eigen values of A are $\lambda = 4, 6$

Eigen values of B are $\lambda = 33, 72$.

Eigen vectors are $x_1 = [1, 1]^T$ $x_2 = [2, 1]^T$

6 For the matrix $A=\begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$ find the eigen values and eigen vectors of 2

the matrix $B = 2A^3 - 3A^2 + 4A - 5I$.

Ans:- Eigen values of A are $\lambda = 1, 2, 3$ Eigen values of B are $\lambda = -2, 7, 34$

Eigen vectors are $X_1 = [1, 0, -1]^T$ $X_2 = [0, 1, 0]^T$ $X_3 = [1, 0, 1]^T$

7 Let a 4×4 matrix A have eigen values $1, -1, 2, -2$. Find the value of the determinant of the matrix $B = 2A + A^{-1} - I$.

Ans:- $\lambda = 2, -4, \frac{7}{2}, -\frac{11}{2}$, $|B| = 154$.

8 Let a 3×3 matrix A have eigen values $1, 2, -1$. Find the trace of the matrix $B = A - A^{-1} + A^2$.

Ans:- $1, \frac{11}{2}, 1$, Trace of $B = \frac{15}{2}$.

Matrix Polynomial :—

An expression of the form $F(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_m x^m$, $A_m \neq 0$.
Where $A_0, A_1, A_2, \dots A_m$ are matrices each of order $n \times n$ over a field F, is called a matrix polynomial of degree m.

The symbol x is called indeterminate and will be assumed that it is commutative with every matrix coefficient.

The matrices themselves are matrix polynomials of zero degree.

Equality of Matrix Polynomials :—

Two matrix polynomials are equal if and only if the coefficients of like powers of x are the same.

The Cayley Hamilton Theorem :—

Every square matrix satisfies its own characteristic equation.

Determination of A^{-1} using Cayley Hamilton Theorem :—

The matrix A satisfies its characteristic equation.

i.e $(-1)^n \left[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I\right] = 0$

$$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0.$$

Multiplying bothsides by A^{-1}, we get.

$$A^{-1}\left[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I\right] = 0$$

$$A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \dots + a_n A^{-1} = 0$$

If A is non singular, then we have.

$$a_n A^{-1} = -A^{n-1} - a_1 A^{n-2} - a_2 A^{n-3} - \dots - a_{n-1} I.$$

$$A^{-1} = \frac{-1}{a_n}\left[A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \dots + a_{n-1} I\right].$$

(1) Find the inverse of the matrix. $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by using cayley Hamilton theorem.

verify Cayley Hamilton theorem. and hence find A^4.

Sol:- Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$.

i.e $$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)\left[(1-\lambda)(2-\lambda)-1\right] + 1\left[0-2\right] = 0.$$

$$(1-\lambda)\left[2-3\lambda+\lambda^2-1\right]-2 = 0$$

$$(1-\lambda)\left[\lambda^2-3\lambda+1\right]-2 = 0$$

$$\lambda^2-3\lambda+1-\lambda^3+3\lambda^2-\lambda-2 = 0$$

$$-\lambda^3+4\lambda^2-4\lambda-1 = 0$$

$$\lambda^3-4\lambda^2+4\lambda+1 = 0.$$

We know that the Cayley hamilton theorem.

Every square matrix satisfies its own characteristic equation.

$$A^3-4A^2+4A+I = 0.$$

Multiply bothsides by A^{-1}, we get

$$A^{-1}(A^3-4A^2+4A+I) = A^{-1}(0)$$

$$A^2-4A+4I+A^{-1} = 0.$$

$$A^{-1} = -A^2+4A-4I.$$

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -2 & -2 & -3 \\ -6 & -1 & -5 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}.$$

Veribication :-

We know that the cayley Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

i.e $A^3 - 4A^2 + 4A + I = 0$.

$$A^3 = A^2 . A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix}$$

$$A^3 - 4A^2 + 4A + I = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - 4\begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} + 4\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 - 4A^2 + 4A + I = 0$

$\therefore$ cayley Hamilton theorem is veribied.

To bind A^4 :-

We have $A^3 - 4A^2 + 4A + I = 0$

Pre multiply with 'A', we get

$$A(A^3 - 4A^2 + 4A + I) = A(0).$$

$$A^4 - 4A^3 + 4A^2 + A = 0$$

$$A^4 = 4A^3 - 4A^2 - A$$

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$$A^4 = 4\begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - 4\begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -9 & -7 & -12 \\ 24 & 3 & 19 \\ 38 & -5 & 22 \end{bmatrix}.$$

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Using Cayley-Hamilton theorem, Express $A^6-4A^5+8A^4-12A^3+14A^2$ as a linear polynomial in A, where $A=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}$ 63

sol:- Given that $A=\begin{bmatrix}1 & 2\\ -1 & 3\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

i.e $\begin{vmatrix}1-\lambda & 2\\ -1 & 3-\lambda\end{vmatrix}=0$

$(1-\lambda)(3-\lambda)+2=0.$

$\lambda^2-4\lambda+5=0.$

We know that The Cayley Hamilton Theorem.

Every square matrix satisfies its own characteristic equation

i.e $A^2-4A+5I=0.$

$A^2=4A-5I.$ —— (1)

Pre multiplying (1) by A, A^2, A^3 and A^4, we get

$A^3=4A^2-5A.$

$A^4=4A^3-5A^2$

$A^5=4A^4-5A^3$

$A^6=4A^5-5A^4.$

$$\begin{aligned}A^6-4A^5+8A^4-12A^3+14A^2 &= 4A^5-5A^4-4A^5+8A^4-12A^3+14A^2\\ &= 3A^4-12A^3+14A^2\\ &= 3(4A^3-5A^2)-12A^3+14A^2\\ &= 12A^3-15A^2-12A^3+14A^2\\ &= -A^2\\ &= 5I-4A.\end{aligned}$$

Which is a linear polynomial in A.

Verify Cayley Hamilton theorem for $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$ and hence find A^4 and A^{-1}.

Hence find the matrix represented by $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$.

sol: Given that $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

i.e $\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 1-\lambda & 0\\ 1 & 1 & 2-\lambda\end{vmatrix}=0$

$$(1-\lambda)\left[(2-\lambda)^2-1\right]=0$$

$$(1-\lambda)(\lambda^2-4\lambda+3)=0$$

$$\lambda^3-5\lambda^2+7\lambda-3=0.$$

Verification:-

We know that cayley Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

i.e $A^3-5A^2+7A-3I=0$.

$$A^2=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{bmatrix}$$

$$A^3=\begin{bmatrix}5 & 4 & 4\\ 0 & 1 & 0\\ 4 & 4 & 5\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}=\begin{bmatrix}14 & 13 & 13\\ 0 & 1 & 0\\ 13 & 13 & 14\end{bmatrix}$$

$$A^3-5A^2+7A-3I = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3-5A^2+7A-3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

$\therefore$ Cayley Hamilton theorem verified.

To find A^4 :-

We have $A^3-5A^2+7A-3I=0$.

Pre multiply with A, we get

$$A(A^3-5A^2+7A-3I) = A(0)$$

$$A^4-5A^3+7A^2-3A = 0$$

$$A^4 = 5A^3-7A^2+3A$$

$$A^4 = 5\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 7\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 3\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

$$A^4 = \begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix}$$

To find A^{-1} :-

We have $A^3 - 5A^2 + 7A - 3I = 0$.

Pre multiply with A^{-1}, we get

$$A^{-1}(A^3 - 5A^2 + 7A - 3I) = A^{-1}(0)$$

$$A^2 - 5A + 7I - 3A^{-1} = 0.$$

$$3A^{-1} = A^2 - 5A + 7I.$$

$$3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$\rightarrow$ We have $A^3 - 5A^2 + 7A - 3I = 0$.

$$A^3 = 5A^2 - 7A + 3I.$$

Pre multiply with 'A', we get -

$$A^4 = 5A^3 - 7A^2 + 3A.$$

$$A^5 = 5A^4 - 7A^3 + 3A^2$$

$$A^6 = 5A^5 - 7A^4 + 3A^3$$

$$A^7 = 5A^6 - 7A^5 + 3A^4$$

$$A^8 = 5A^7 - 7A^6 + 3A^5.$$

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$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I.$$

$$=(5A^7-7A^6+3A^5)-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$$

$$=A^4-5A^3+8A^2-2A+I.$$

$$=(5A^3-7A^2+3A)-5A^3+8A^2-2A+I.$$

$$=A^2+A+I.$$

$$\therefore A^2+A+I=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}.$$

• If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$. Hence find A^{50}.

(31)

Sol:- Given that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 1) = 0.$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

We know that Cayley Hamilton theorem.

Every square matrix satisfies its own characteristic equation.

$$A^3 - A^2 - A + I = 0.$$

$$A^3 - A^2 = A - I.$$

Pre multiplying both sides successively by A, we obtain.

$$A^3 - A^2 = A - I.$$

$$A^4 - A^3 = A^2 - A$$

$$A^5 - A^4 = A^3 - A^2$$

$$\cdots\cdots\cdots$$

$$A^{n-1} - A^{n-2} = A^{n-3} - A^{n-4}$$

$$A^n - A^{n-1} = A^{n-2} - A^{n-3}$$

Adding these equations, we get

$$A^n - A^2 = A^{n-2} - I.$$

$$A^n = A^{n-2} + A^2 - I, \quad n \geq 3.$$

Using this equation recursively, we get

$$A^{n-2} = A^{(n-2)-2} + A^2 - I = A^{n-4} + A^2 - I.$$

$$A^n = (A^{n-4} + A^2 - I) + A^2 - I.$$

$$A^n = A^{n-4} + 2(A^2 - I)$$

$$= (A^{n-6} + A^2 - I) + 2(A^2 - I) = A^{n-6} + 3(A^2 - I)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= A^{n-(n-2)} + \frac{1}{2}(n-2).(A^2 - I).$$

$$= \frac{n}{2}A^2 - \frac{1}{2}(n-2)I.$$

substituting $n=50$, we get

$$A^{50} = 25A^2 - 24I = 25\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} - 24\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\therefore A^{50} = \begin{bmatrix}1 & 0 & 0\\ 25 & 1 & 0\\ 25 & 0 & 1\end{bmatrix}.$$

CAYLEY- HAMILTON THEOREM

1 State Cayley Hamilton theorem

2 Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Hence find (i) A^{-1} (ii) A^4.

Ans:- $A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$.

3 Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$.

Hence find (i) A^4 and show that (ii) $A^3 = -9A$ (iii) $A^5 = 81A$.

4 Prove that the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ satisfies its characteristic equation

Using C.H.T. show that (i) $A^4 = I$ and (ii) $A^3 = A^{-1}$. Also Find A^4.

5 If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ find A^8 using the cayley Hamilton theorem.

6 Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

Express $A^4 - 3A^3 + 2A^2 - 5I$ as a linear polynomial in A.

7 If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. Verify Cayley Hamilton theorem for the matrix A.

Hence find (i) A^4 (ii) A^{-1}. Also Find the matrix

$B = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ Ans: $A^2 + A + I$

8 If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ show that $A^8 - 4A^5 + 8A^4 - 12A^3 + 14A^2 = \begin{bmatrix} 1 & -8 \\ 4 & -7 \end{bmatrix}$.

9 For the matrix $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ Express A^3, A^4 and A^{-1} in terms of I, A and A^2

by using the cayley Hamilton Theorem. Hence find these explicitely.

Ans:- $A^3 = \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$ $A^4 = \begin{bmatrix} 1146 & -1904 & 1226 \\ 322 & -639 & 476 \\ 359 & -544 & 407 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 9 & 0 & -22 \\ 10 & -4 & -24 \\ 7 & -8 & -10 \end{bmatrix}\frac{1}{22}$

10 If $A=\begin{bmatrix}1 & 3 & 7\\ 4 & 2 & 3\\ 1 & 1 & 1\end{bmatrix}$ find A^3, A^4 and A^{-2} by using Cayley Hamilton theorem.

Ans:- $A^3=\begin{bmatrix}135 & 152\\ 140 & 163\\ 60 & 76\end{bmatrix}$ $A^4=\begin{bmatrix}935 & 1173 & 1633\\ 1000 & 1162 & 1677\\ 475 & 554 & 759\end{bmatrix}$ $A^{-2}=\frac{1}{245}\begin{bmatrix}-5 & -23 & 69\\ 32 & 10 & -79\\ -17 & 10 & 19\end{bmatrix}$

11 Find the characteristic equation of the matrix $A=\begin{bmatrix}3 & 1 & 1\\ -1 & 5 & -1\\ 1 & 1 & 5\end{bmatrix}$ and show that it is satisfied by A and hence obtain its A^{-1}.

Ans:- $A^{-1}=\frac{1}{72}\begin{bmatrix}24 & -6 & -6\\ 4 & 14 & 2\\ -4 & 4 & 16\end{bmatrix}$

12 Find the characteristic equation of the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}$ and show that it is satisfied by A and hence obtain its A^{-1}.

Ans:- $A^{-1}=\begin{bmatrix}1 & -3 & 2\\ -3 & 3 & -1\\ 2 & -1 & 0\end{bmatrix}$

13 Using Cayley Hamilton theorem, express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A. Where $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$ Ans:- $A+5I$.

14 Show that the matrix $A=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$ satisfies the Cayley Hamilton theorem. Hence find A^4.

15 If $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$ show by using the Cayley-Hamilton theorem that

(i) $A^4=2A^2-I$ (ii) $A^5=2A^4+A-2I$.

DIAGONALIZATION OF A MATRIX :-

Let A be a square matrix of order n. Then A is said to be diagonalizable if there exists a matrix P of order n such that $P^{-1}AP = D$ where D is a diagonal matrix. Then $P^{-1}AP$ is a diagonal form of A.

P is formed by the linearly independent eigen vectors corresponding to the eigen values of A then $P = [x_1\ x_2\ x_3 \dots x_n]$ is said to be transforming matrix of A and it reduces the matrix A to the diagonal form D.

Similarity of Matrices :-

Let A and B are square matrices of order n. Then B is said to be similar to A if there exists a non singular matrix P such that $B = P^{-1}AP$.

Algebraic and Geometric multiplicities of a characteristic root :-

If λ be a characteristic root of a order 't' of the characteristic equation $|A-\lambda I|=0$, then t is called the "algebraic multiplicity" of λ i.e the order of the characteristic λ, is said to be "algebraic multiplicty". It is denoted by "t".

If s is the number of linearly independent Eigen vectors corresponding to the Eigen value λ, then 's' is called the "geometric multiplicity" of λ i.e The number of linearly independent Eigen vectors corresponding to the Eigen value λ, is said to be its geometric multiplicity. It is denoted by 's'.

The geometric multiplicity of a characteristic root cannot exceed its algebraic multiplicity i.e $s \le t$.

Note :-

(2)

(i) If A is similar to a diagonal matrix B then the diagonal elements of B are the eigen values of A.

(ii) If A is a square matrix of order n is diagonalizable iff it possesses n linearly independent eigen vectors.

(iii) If the Eigen values of an $n \times n$ matrix are all distinct, then it is always similar to a diagonal matrix i.e a diagonalizable matrix

(iv) If the Eigen values of a matrix are not distinct, then we have to verify the following condition or test for the diagonalization of a matrix.

Condition for the diagonalization :-

The necessary and sufficient condition for a square matrix A to be diagonalizable is that the geometric multiplicity of each of its Eigen values coincides with the algebraic multiplicity.

Modal and Spectral Matrices :-

If a square matrix A is diagonalizable then the matrix P which transforms A to the diagonal form D is called the modal matrix of A and the matrix D is called the spectral matrix of A.

Let x_1, x_2, x_3 are Eigen vectors corresponding to the Eigen values $\lambda_1, \lambda_2, \lambda_3$ of A respectively then the modal matrix of A is $P = [x_1 \ x_2 \ x_3]$ and the spectral matrix of A is $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

exists such that $P^{-1}AP = D$.

Working procedure to Diagonalize a square matrix A :-

Let the square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Step 1 :- The characteristic equation of A is $|A-\lambda I| = 0$

i.e $\begin{vmatrix} a_{11}-\lambda & a_{21} & a_{31} \\ a_{21} & a_{22}-\lambda & a_{32} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$

Step 2 :- Solve the characteristic equation and find the Eigen values $\lambda_1, \lambda_2, \lambda_3$ of the given matrix A.

Step 3 :-

Case (i) :- The Eigen values of matrix A are distinct.

(a) Find the Eigen vectors X_1, X_2, X_3 corresponding to the Eigen values λ_1, λ_2 and λ_3.

Let $X_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ $X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ $X_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$

(b) Consider the Modal Matrix $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

(c) Find $P^{-1} = \frac{1}{|P|} \text{Adj } P$.

(d) Find $P^{-1}AP$ which is the diagonal matrix of A.

$$P^{-1}AP = D = \text{Diag}[\lambda_1 \ \lambda_2 \ \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Case (ii) :- The Eigen values of matrix A are not distinct.

Suppose $\lambda_1 = \lambda_2$ and λ_3 is distinct.

Here algebraic multiplicity of $\lambda_1 = 2$ and algebraic multiplicity of $\lambda_3 = 1$.

(a) Find the Eigen vectors corresponding to the Eigen values λ_1, λ_2 and λ_3

Let X_1, X_2 are the Eigen vectors corresponding to the Eigen value λ_1 and X_3 is the Eigen vector corresponding to the Eigen value λ_3.

Let $X_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ $X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ $X_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$ (4)

Geometric multiplicity of $\lambda_1 = 2$, Geometric multiplicity of $\lambda_3 = 1$.

$\therefore$ Algebraic multiplicity of λ_1 = Geometric multiplicity of $\lambda_1 = 2$

Algebraic multiplicity of λ_3 = Geometric multiplicity of $\lambda_3 = 1$

(b) Consider the Modal matrix $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

(c) Find $P^{-1} = \frac{1}{|P|} \text{Adj} P$.

(d) Find $P^{-1}AP$ which is the diagonal matrix of A.

$\therefore P^{-1}AP = D = \text{Diag}[\lambda_1 \; \lambda_2 \; \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

<u>Case (iii)</u> :- The Eigen values of matrix A are not distinct.

Suppose $\lambda_1 = \lambda_2$ and λ_3 is distinct.

Here algebraic multiplicity of $\lambda_1 = 2$, algebraic multiplicity of $\lambda_3 = 1$.

(a) Find the Eigen vectors corresponding to the Eigen values $\lambda_1, \lambda_2, \lambda_3$.

Let X_1 be the Eigen vector corresponding to Eigen value λ_1

Let X_3 be the Eigen vector corresponding to Eigen value λ_3.

Here Algebraic multiplicity of $\lambda_1 \neq$ Geometric multiplicity of λ_1

$\therefore$ A is not diagonalizable.

Computation of positive integral powers of matrix A :-

Let A be a square matrix of order 3. Then there exists a non singular matrix P such that $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Where $\lambda_1, \lambda_2, \lambda_3$ are Eigen values of A.

$$P^{-1}AP = D$$

$$(P^{-1}AP)^2 = D^2$$

$$(P^{-1}AP)(P^{-1}AP) = D^2$$

$$P^{-1}A(PP^{-1})AP = D^2$$

$$P^{-1}AIAP = D^2$$

$$P^{-1}A^2P = D^2$$

Similarly $P^{-1}A^3P = D^3$

$$\cdots$$

$$P^{-1}A^nP = D^n \quad \text{——} (1)$$

Now pre multiplying the eqn (1) with P and post multiplying with P^{-1}

we have $P(P^{-1}A^nP)P^{-1} = PD^nP^{-1}$

$$(PP^{-1})A^n(PP^{-1}) = PD^nP^{-1}$$

$$IA^nI = PD^nP^{-1}$$

$$A^n = PD^nP^{-1}$$

$\therefore A^n = PD^nP^{-1}$ Where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$

→ Find the matrix P which transforms the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to the diagonal form. Hence evaluate A^4.

Sol:- Given that $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ and λ is an eigen value of A.

The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\cdot\left[(2-\lambda)(3-\lambda) - 2\right] - \left[2 - 2(2-\lambda)\right] = 0.$$

$$(1-\lambda)\left(6 - 5\lambda + \lambda^2 - 2\right) - 2\lambda + 2 = 0.$$

$$\lambda^2 - 5\lambda + 4 - \lambda^3 + 5\lambda^2 - 4\lambda - 2\lambda + 2 = 0.$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \text{--- (1)}.$$

$\lambda = 1$ is one of the roots of the equation (1).

$$\begin{array}{c|cccc} \lambda=1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\lambda = 1, 2, 3.$$

∴ Eigen values of A are $\lambda = 1, 2, 3$.

The Eigen values of A are distinct.

∴ The matrix A is diagonalizable.

Now we have to find Eigen vectors $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponding to Eigen value λ are obtained by solving the homogeneous system

$$(A - \lambda I)X = 0$$

i.e $\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — (2).

Case(i):- Eigen vector corresponding to Eigen value $\lambda = 1$:-

For $\lambda = 1$, The system (1) can be written as .

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by applying E-row operations only and hence determine rank of coeff. matrix.

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Which is in echelon form .

Here rank of the coefficient matrix of the system is 2 i.e $r=2$

Sothat the system has $n-r = 3-2 = 1$ L.I solution .

There is only one L.I eigen vector corresponding to the Eigen value $\lambda = 1$.

To determine this we have to assign an arbitrary value for $n-r = 3-2 = 1$ variable .

From the above system, the equations can be written as

$$x_1 + x_2 + x_3 = 0$$

$$-x_3 = 0 \implies x_3 = 0 .$$

To determine this we have to assign an arbitrary value for $n-r = 3-2 = 1$ variable.

From the above system, the eqn's can be written as.

$$-x_1 - x_3 = 0 \Rightarrow x_1 + x_3 = 0$$

$$2x_2 - x_3 = 0$$

choose $x_2 = K_2$

$$x_3 = 2x_2$$

$$x_3 = 2K_2$$

$$x_1 = -x_3 = -2K_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2K_2 \\ K_2 \\ 2K_2 \end{bmatrix} = K_2 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \text{ where } K_2 \neq 0$$

$X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 2$.

Case (iii) Eigen vector corresponding to the eigen value $\lambda = 3$:-

For $\lambda = 3$ The system ② can be written as

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by applying E-row operations only. and hence determine the coeff. matrix.

$R_2 \to 2R_2 + R_1 \qquad R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

$$x_1 + x_2 = 0 \qquad [\because x_3 = 0]$$

choose $x_2 = k_1$

$$x_1 = -x_2 = -k_1.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ where } k_1 \neq 0.$$

$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$.

Case (ii) :- Eigen vector corresponding to the Eigen value $\lambda = 2$:-

For $\lambda = 2$, The system ② can be written as

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by applying E-row operations only and hence determine rank of coeff. matrix.

$$R_2 \to R_2 + R_1, \quad R_3 \to R_3 + 2R_1$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the co.eff. matrix of the system is 2 i.e $r = 2$.

Sothat the system has $n - r = 3 - 2 = 1$ L.I solution.

There is only one L.I eigen vector corresponding to the eigen value $\lambda = 2$.

Here the rank of the coeff. matrix of the system is 2. i.e $r=2$

The system has $n-r=3-2=1$ L.I solution.

There is only one linearly independent eigen vector corresponding to the eigen value $\lambda=3$.

To determine this we have to assign an arbitrary value for $n-r=3-2=1$ variable.

From the above system, the eqn's can be written as

$$-2x_1-x_3=0 \Rightarrow 2x_1+x_3=0$$
$$-2x_2+x_3=0 \Rightarrow 2x_2-x_3=0$$

choose $x_1=k_3$ Then $x_3=-2k_3$

$2x_2=x_3$ Then $2x_2=-2k_3$

$x_2=-k_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_3 \\ -k_3 \\ 2k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ where } k_3 \neq 0.$$

$X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda=3$.

Consider $P=[X_1 \ X_2 \ X_3] = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$

Which is the modal matrix.

$$|P| = -1(-2+2)+2(-2-0)+1(2-0)$$
$$|P| = -2$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P = \frac{-1}{2} \begin{bmatrix} 0 & -2 & 1 \\ 2 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ -2 & 2 & -1 \end{bmatrix}$$

Thus the matrix P transforms the matrix A to the diagonal form which is given by $P^{-1}AP = D$.

$$P^{-1}A = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ -2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -4 & -4 & 0 \\ -6 & -6 & -3 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -4 & -4 & 0 \\ -6 & -6 & -3 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \text{Diag}(1,2,3) = D.$$

Hence $P^{-1}AP$ is a diagonal matrix.

where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is the spectral matrix.

To find A^4 :-

We have $A^n = PD^nP^{-1}$

$n=4$, $A^4 = PD^4P^{-1}$

$$A^4 = \frac{1}{2}\begin{bmatrix} -1 & -2 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}.$$

→ Diagonalize the matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$ and Hence find A^4.

Sol. Given that $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix}$ and λ is an eigen value of A

The characteristic eqn. of A is $|A - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} = 0$$

$C_2 \to C_2 - C_1, \quad C_3 \to C_3 - C_1$

$$(2-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda)^2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda)^2 = 0$$

$$\lambda = 1, 1, 2.$$

$\therefore$ Eigen values of A are $\lambda = 1, 1, 2$.

The Algebraic multiplicities of each eigen values 1 and 2 are 2 and 1.

Now the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corresponding to the eigen value λ are obtained by solving the homogeneous system $(A-\lambda I)X = 0$

i.e $\begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — ①

Case (i) :- Eigen vector corresponding to the Eigen value $\lambda = 1$:-

For $\lambda = 1$, The system ① can be written as

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by using E-row operations only.

$R_2 \to 2R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the coeff- matrix of the system is 1 i.e $r = 1$

So that the system has $n - r = 3 - 1 = 2$ L.I solutions.

There are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 1$.

To determine this we have to assign an arbitrary value for $n - r = 3 - 1 = 2$ variable.

From the above system, the eqn's can be written as

$2x_1 + 2x_2 + 2x_3 = 0$ i.e $x_1 + x_2 + x_3 = 0$

choose $x_2 = K_1$, $x_3 = K_2$

$x_1 = -x_2 - x_3 = -K_1 - K_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K_1 - K_2 \\ K_1 \\ K_2 \end{bmatrix} = K_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent eigen vectors.

Corresponding to the eigen value $\lambda = 1$.

The geometric multiplicity of the eigen value $\lambda = 1$ is 2.

Case (ii) Eigen vector corresponding to the Eigen value $\lambda = 2$:-

For $\lambda = 2$, The system ① can be written as

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ -2 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reduce the coeff. matrix into echelon form by applying E-row operations only and hence determine the rank of coeff. matrix

$$R_2 \to R_2 - R_1, \; R_3 \to R_3 + 2R_1$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is in echelon form.

Here the rank of the coeff. matrix of the system is 2 i.e $r = 2$

Sothat the system has $n - r = 3 - 2 = 1$ linearly independent sol.

There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = 2$.

To determine this we have to assign an arbitrary value for $n - r = 3 - 2 = 1$ variable.

From the above system, The equations can be written as

$$x_1 + 2x_2 + 2x_3 = 0$$

$$-2x_2 - x_3 = 0 \Rightarrow 2x_2 + x_3 = 0$$

choose $x_2 = k_3$

$$x_3 = -2x_2 = -2k_3$$

$$x_1 = -2x_2 - 2x_3$$

$$x_1 = -2k_3 + 4k_3$$

$$x_1 = 2k_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_3 \\ k_3 \\ -2k_3 \end{bmatrix} = k_3 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ where } k_3 \neq 0.$$

$X_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is the L.I eigen vector corresponding to the eigen value $\lambda = 2$.

The geometric multiplicity of the eigen value $\lambda = 2$ is 1.

Since the geometric multiplicity of each eigen value of A coincides with the algebraic multiplicity.

$\therefore$ A is diagonalizable matrix.

The modal matrix $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

$$|P| = \begin{vmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{vmatrix}$$

$$|P| = -1(0-1) + 1(-2-0) + 2(1-0)$$

$$|P| = 1$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P.$$

Cofactor matrix of $P = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

$$\operatorname{Adj} P = [\text{Cofactor matrix of } P]^T = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus the matrix P transforms the matrix A to the diagonal form which is given by $P^{-1}AP = D$

$$P^{-1}A = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D = \mathrm{Diag}(1, 1, 2)$$

Hence $P^{-1}AP$ is diagonal matrix.

Where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is the spectral matrix.

<u>To find A^4 :-</u>

We have $A^n = P D^n P^{-1}$

$n = 4$, $A^4 = P D^4 P^{-1}$

$$A^4 = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 32 \\ 1 & 0 & 16 \\ 0 & 1 & -32 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 31 & 30 & 30 \\ 15 & 16 & 15 \\ -30 & -30 & -29 \end{bmatrix}$$

Find an orthogonal matrix that will diagonalize the real Symmetric matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. Also find the resulting diagonal matrix.

Sol:- Given that $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$.

$$\text{i.e} \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 + R_3$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 2-\lambda & 2-\lambda \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \begin{vmatrix} 6-\lambda & -2 & 2 \\ 0 & 1 & 1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$C_3 \to C_3 - C_2$

$$(2-\lambda) \begin{vmatrix} 6-\lambda & -2 & 4 \\ 0 & 1 & 0 \\ 2 & -1 & 4-\lambda \end{vmatrix} = 0.$$

Expanding by R_2, we have.

$$(2-\lambda)\left[(6-\lambda)(4-\lambda)-8\right] = 0$$

$$(2-\lambda)(\lambda^2 - 10\lambda + 16) = 0.$$

$$(2-\lambda)(\lambda-2)(\lambda-8) = 0$$

$$\lambda = 2, 2, 8.$$

$\therefore$ The Eigen values of A are 2, 2, 8. Which are not distinct.

The algebraic multiplicities of each eigen values 2 and 8 are 2 and 1.

Now the Eigen vectors corresponding to the Eigen values λ are obtained by solving the system of equations $(A-\lambda I)X = 0$ —— ①.

Case(i):- Eigen vector corresponding to the Eigen value $\lambda = 2$:-

For $\lambda = 2$ The system ① can be written as

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coeff. matrix into echelon form by applying E-row operations only.

$R_2 \rightarrow 2R_2 + R_1 \quad R_3 \rightarrow 2R_3 - R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Rank of the co efficient matrix of the system is 2 i.e $r=2$.

There is so that the homogeneous system has $n-r = 3-2=1$ linearly independent solution. corresponding to the eigen value $\lambda = 2$.

There is only one linearly independent eigen vector.

To determine this ; From the above system

The equations can be written as

$$4x_1 - 2x_2 + 2x_3 = 0.$$

$$2x_1 - x_2 + x_3 = 0.$$

choose $x_1 = k_1 \quad x_2 = k_2$

$$x_3 = x_2 - 2x_1$$

$$x_3 = k_2 - 2k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_2 - 2k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 2$.

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So that the geometric multiplicity of the eigen value $\lambda = 2$ is 2.

Case (iii):- Eigen Vector corresponding to the Eigen value $\lambda = 8$.

For $\lambda = 8$, The system ① can be written as.

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the Coeff. matrix into echelon form by applying E-row operations only.

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$ = The No. of Non zero rows equivalent to A.

$\rho(A) = 2 < 3$ (No. of non zero rows)

So that the homogeneous system have $n-r = 3-2 = 1$ linearly independent solutions.

There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = 8$.

To determine this, we have to assign an arbitrary value for one variable.

From the above system the linear equations are

$$x_1 + x_2 - x_3 = 0$$

$$x_2 + x_3 = 0$$

choose $x_3 = k_3$

$$x_2 = -x_3 = -k_3$$

$$x_1 = x_3 - x_2 = +k_3 + k_3 = 2k_3$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_3 \\ -k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corrosponding to the eigen value $\lambda = 8$.

so that the geometric multiplicity of $\lambda = 8$ is 1.

since the geometric multiplicity of each eigen value of A coincides with the algebraic multiplicity

$\therefore$ A is diagonalizable matrix.

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are the eigen vectors corrosponding to the eigen values $\lambda = 2, 2, 8$.

Here the eigen vectors X_1 and X_2 are not pairwise orthogonal.

Now we have to find the another linearly independent eigen vector X_2 pairwise orthogonal to X_1 and X_3.

Let $X_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the another linearly independent eigenvector. corrosponding to $\lambda = 2$ and is orthogonal to X_1 and X_3.

x_1, x_2 are pairwise orthogonal if $a + 0.b - 2c = 0$.

x_2, x_3 are pairwise orthogonal if $2a - b + c = 0$.

solving above two equations, we get

$$\frac{a}{-2} = \frac{b}{-5} = \frac{c}{-1}$$

$$\begin{matrix} 0 & -2 & 1 & 0 \\ -1 & 1 & 2 & -1 \end{matrix}$$

$a = -2 \quad b = -5 \quad c = -1$

$\therefore$ The Eigen vectors $x_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $x_2 = \begin{bmatrix} -2 \\ -5 \\ -1 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are pairwise orthogonal.

Consider the modal matrix $[x_1 \ x_2 \ x_3] = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -5 & -1 \\ -2 & -1 & 1 \end{bmatrix}$

$$\|x_1\| = \sqrt{1+0+4} = \sqrt{5} \qquad \|x_2\| = \sqrt{4+25+1} = \sqrt{30}$$

$$\|x_3\| = \sqrt{4+1+1} = \sqrt{6}.$$

Normalized modal matrix. $P = \left[\frac{x_1}{\|x_1\|} \ \frac{x_2}{\|x_2\|} \ \frac{x_3}{\|x_3\|}\right]$

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{-5}{\sqrt{30}} & \frac{-1}{\sqrt{6}} \\ \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Which is the orthogonal matrix.

By definition $PP^T = P^TP = I \implies P^{-1} = P^T$

The matrix P will reduce the matrix A to the diagonal form which is given by $P^{-1}AP = D$ i.e $P^TAP = D$.

$$P^T A P = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \\ \frac{-2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & \frac{-1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{-5}{\sqrt{30}} & \frac{-1}{\sqrt{6}} \\ \frac{-2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = D$$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is the spectral matrix .

DIAGONALIZATION OF A MATRIX.

1 Define Modal matrix

2 Define Spectral matrix

3 Define Similarity of matrices.

4 Explain Diagonalization of a square matrix.

2:

1 show that the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ is diagonalizable. Hence find P such that $P^{-1}AP$ is a diagonal matrix. Then, obtain the matrix $B = A^2+5A+3I$. Ans:- $\lambda = 1, 2, 3$; $x_1 = [1 \ -1 \ 1]^T$, $x_2 = [1 \ 0 \ 1]^T$

$x_3 = [0 \ 1 \ 1]^T$ $P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 25 & 8 & -8 \\ -18 & 9 & 18 \\ -2 & 8 & 19 \end{bmatrix}$

2 show that the matrix $A = \begin{bmatrix} -3 & -2 & 1 \\ -2 & 0 & 4 \\ -6 & -3 & 5 \end{bmatrix}$ is diagonalizable. Find the matrix P such that $P^{-1}AP$ is a diagonal matrix.

Ans:- $\lambda = 1, x_1 = [1, -2, 0]^T$; $\lambda = -1$, $x_2 = [3, -2, 2]^T$; $\lambda = 2$. $x_3 = [-1, 3, 1]^T$

$P = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ $P^{-1} = \frac{1}{2}\begin{bmatrix} -8 & -5 & 7 \\ 2 & 1 & -1 \\ 4 & -2 & 4 \end{bmatrix}$

3 show that the matrix $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix}$ is diagonalizable. Find the matrix P such that $P^{-1}AP$ is a diagonal matrix.

Ans:- $\lambda = 0$, $x_1 = [3, 1, -2]^T$; $\lambda = 2i$, $x_2 = [3+i, 1+3i, -4]^T$; $\lambda = -2i$

$x_3 = [3-i, 1-3i, -4]^T$

$P = \begin{bmatrix} 3 & 3+i & 3-i \\ 1 & 1+3i & 1-3i \\ -2 & -4 & -4 \end{bmatrix}$ $P^{-1} = \frac{1}{32}\begin{bmatrix} 24 & -8 & 16 \\ 2i-6 & 2-6i & -8 \\ -2i-6 & 2+6i & 8 \end{bmatrix}$.

R.NO —— Q.NO

C1 – F0 —— 1, 3, 5, 7, 9, 11, 13, 15,

F1 – J0 —— 2, 4, 6, 8, 10, 12, 14, 16.

4 Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ Hence determine A^4. 6

Ans:- $\lambda = 0$, $x_1 = [1, 0, -1]^T$; $\lambda = 1$, $x_2 = [-1, -1, 1]^T$; $\lambda = 2$, $x_3 = [1, 1, 0]^T$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

5 Diagonalize the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ Hence determine A^3

Ans:- $\lambda = 1$, $x_1 = [1, -1, -1]^T$; $\lambda = 2$, $x_2 = [0, 1, 1]^T$; $\lambda = -2$; $x_3 = [8, -5, 7]^T$

$$P = \begin{bmatrix} 1 & 0 & 8 \\ -1 & 1 & -5 \\ -1 & 1 & 7 \end{bmatrix} \quad P^{-1} = \frac{1}{12}\begin{bmatrix} 12 & 8 & -8 \\ 12 & 15 & -3 \\ 0 & -1 & 1 \end{bmatrix}$$

6 Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ Hence determine A^5.

Ans:- $\lambda = 1$, $x_1 = [+1, +2, -2]^T$; $\lambda = 2$, $x_2 = [1, 1, 0]^T$; $\lambda = 3$, $x_3 = [1, 1, 1]^T$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} \quad A^5 = \begin{bmatrix} -359 & 391 & 211 \\ -360 & 392 & 211 \\ -484 & 484 & 243 \end{bmatrix}$$

7 Diagonalize the matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ Hence determine A^4.

Ans:- $\lambda = 2, 2$, $x_1 = [1, 0, -1]^T$ $x_2 = [-2, 1, 0]^T$ $\lambda = 4$, $x_3 = [1, 0, 1]^T$

$$P = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}\frac{1}{2}$$

8 Diagonalize the matrix $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ Hence determine A^6

Ans:- $\lambda = 1$, $x_1 = [3, -1, 3]^T$; $\lambda = 2, 2$ $x_2 = [2, 0, 1]^T$ $x_3 = [2, 1, 0]^T$.

$$P = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & -5 \\ -1 & 3 & 2 \end{bmatrix}$$

9. Diagonalize the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ Hence determine A^3.

$\lambda = 1, 1 \quad x_1 = [1, 0, -1]^T \; x_2 = [0, 1, -2]^T; \quad \lambda = 5, \; x_3 = [1, 1, 1]^T$

$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

10. Diagonalize the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ Hence determine A^4

Ans:- $\lambda = 5, \; x_1 = [1, 2, -1]^T; \; \lambda = -3, -3, \; x_2 = [-2, 1, 0]^T \; x_3 = [3, 0, 1]$

$P = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

11. Diagonalize the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ Hence determine A^5

Ans:- $\lambda = 2, 2, \; x_1 = [1, 2, 0]^T \; x_2 = [-1, 0, 2]^T; \; \lambda = 8, \; x_3 = [2, -1, 1]^T$.

$P = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

12. Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ Hence determine A^4

Ans:- $\lambda = -1, -1, \; x_1 = [1, 0 -1]^T \; x_2 = [0, 1, -1]^T; \; \lambda = 2, \; x_3 = [1, 1, 1]^T$.

13. Diagonalize, if possible $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

Ans:- $\lambda = 1 \; x_1 = [1, 1, -1]^T \; \lambda = 2, 2, \; x_2 = [2, 1, 0]^T$ Not diagonalizable.

14. Diagonalize if possible $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Ans: $\lambda = 1, 1, 1 \quad x = [0, 3, -2]^T$, Not diagonalizable.

15 Diagonalize if possible $A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ -1 & 3 & 4\end{bmatrix}$

Ans:- $\lambda=1,1,\ x_1=[0,1,-1]^T\ \lambda=7,\ x_2=[6,7,5]^T$ Not diagonalizable.

16 Diagonalize if possible $A=\begin{bmatrix}1 & 1 & 1\\ -1 & -3 & -3\\ 2 & 4 & 4\end{bmatrix}$

Ans:- $\lambda=0,0\ \ x_1=[0,1,-1]^T\ \lambda=2\ \ x_2=[1,-2,3]^T$ Not diagonalizable.

17 Diagonalize the matrix $A=\begin{bmatrix}1 & 1 & i\\ 1 & 0 & i\\ -i & -i & 1\end{bmatrix}$ Hence determine A^7

Ans:- $\lambda=0,\ x_1=[1,0,-i]^T;\ \lambda=1+\sqrt{3},\ x_2=[1,\sqrt{3}-1,-i]^T$

$\lambda=1-\sqrt{3}\ \ x_3=[1,-(\sqrt{3}+1),-i]^T$

$P=\begin{bmatrix}i & 1 & 1\\ 0 & \sqrt{3}-1 & -1-\sqrt{3}\\ -1 & -i & -i\end{bmatrix}\quad P^{-1}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

18 Diagonalize the matrix $A=\begin{bmatrix}1 & -1 & 1\\ 1 & 0 & 0\\ -1 & 1 & -1\end{bmatrix}$ Hence determine A^4

Ans:- $\lambda=0,\ x_1=[0,1,1]^T;\ \lambda=i,\ \ x_2=[1,-i,-1]^T;\ \lambda=-i,\ x_3=[1,i,-1]^T$

$P=\begin{bmatrix}0 & 1 & 1\\ 1 & -i & i\\ 1 & -1 & -1\end{bmatrix}\quad P^{-1}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

19 Diagonalize the matrix $A=\begin{bmatrix}-1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$ Hence determine A^5

Ans:- $\lambda=1,\ x_1=[1,0,-1]^T;\ \lambda=\sqrt{5},\ x_2=[\sqrt{5}-1,1,-1]^T;\ \lambda=-\sqrt{5},\ x_3=[\sqrt{5}+1,-1,1]^T$

$P=\begin{bmatrix}1 & \sqrt{5}-1 & \sqrt{5}+1\\ 0 & 1 & -1\\ -1 & -1 & 1\end{bmatrix}\quad P^{-1}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

20 Diagonalize the matrix $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$ Hence determine A^6

Ans:- $\lambda=1,1,\ x_1=[1,1,0]^T\ x_2=[1,0,1]^T;\ \lambda=-2,\ x_3=[-1,1,1]^T$

$P=\begin{bmatrix}1 & 1 & -1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}\quad P^{-1}=\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

→ Find the matrix A whose eigen values are 1, −1, 2 and corresponding eigen vectors are $[1\ 1\ 0]^T$, $[1\ 0\ 1]^T$ and $[3\ 1\ 1]^T$.

Sol: Given eigen values of A are $\lambda_1 = 1$, $\lambda_2 = -1$ $\lambda_3 = 2$

$$\text{spectral matrix } D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$\text{Modal matrix } P = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(0-1) - 1(1-0) + 3(1-0)$$

$$|P| = 1$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P$$

$$P^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

We have $A^n = P D^n P^{-1}$

$$A = P D P^{-1}$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & -4 \end{bmatrix}$$

DIAGONALIZATION OF A MATRIX

1 Find the matrix A whose eigen values and corresponding eigen vectors are as given below.

(a) Eigen values 2, 2, 4; Eigen vectors $[-2,1,0]^T$, $[-1,0,1]^T$, $[1,0,1]^T$

Ans:- $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

2

(b) Eigen values 1, −1, 2; Eigen vectors $[1,1,0]^T$, $[1,0,1]^T$, $[3,1,1]^T$

Ans:- $A = \begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & -4 \end{bmatrix}$

(c) Eigen values 1, 2, 3; Eigen vectors $[1,2,1]^T$, $[2,3,4]^T$, $[1,4,9]^T$

Ans:- $A = \frac{1}{12}\begin{bmatrix} 30 & -12 & 6 \\ 2 & 4 & 14 \\ -34 & 4 & 38 \end{bmatrix}$

(d) Eigen values 0, −1, 1; Eigen vectors $[-1,1,0]^T$, $[1,0,-1]^T$, $[1,1,1]^T$

Ans:- $A = \frac{1}{3}\begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

(e) Eigen values 0, 0, 3; Eigen vectors $[1,2,-1]^T$, $[-2,1,0]^T$, $[3,0,1]^T$

Ans:- $A = \frac{1}{8}\begin{bmatrix} 9 & 18 & 45 \\ 0 & 0 & 0 \\ 3 & 6 & 15 \end{bmatrix}$

(f) Eigen values 1, 1, 3; Eigen vectors $[1,0,-1]^T$, $[0,1,-1]^T$, $[1,1,0]^T$.

Ans:- $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

R.NO —— Q.NO

C1 − E0 —— a, d

E1 − G0 —— b, e

G1 − J0 —— c, f

Singular Value Decomposition :—

Given an $m \times n$ complex matrix A, there in general exist an $m \times m$ unitary matrix U, an $n \times n$ unitary matrix V and an $m \times n$ matrix $D = [d_{ij}]$ with $d_{ij} = 0$ for $i \neq j$ such that $A = UDV^*$ —— ①.

The representation of A as a product of U, D and V^* as given by expression ① is known as the singular value Decomposition (or factorization) of A. The elements d_{ii} in the matrix D are called the singular values of A, the columns of U are called the left singular vectors and the columns of V are called the right singular vectors.

When A is a real matrix, the matrices U and V are orthogonal matrices and D is a real matrix.

In this case, the expression ① becomes. $A = UDV^{-1} = UDV^T$ —— ②.

This expression is equivalent to the expression

$$D = U^{-1}AV = U^TAV \text{ —— ③.}$$

When U and V are known, this expression may be employed to obtain D.

Working Procedure :—

Step 1 : Given the matrix A, obtain the matrices $B = AA^T$ and $C = A^TA$.

Step 2 : obtain the eigen values and corresponding eigen vectors of B. Deduce an orthonormal system from these eigen vectors. Form the orthogonal matrix whose columns are the vectors of this orthonormal system. Denote this orthogonal matrix by U.

Step 3 :- Proceed as in step 2 for the matrix C and obtain the orthogonal matrix V.

Step 4 :- obtain the matrix D by using $D = U^TAV$.

Step 5 :- with U, V and D as determined above, write down the singular value decomposition of A as $A = UDV^T$.

Note :- In the singular value decomposition obtained by the above-mentioned working rule, the elements of D would be such that, for each i, the element d_{ii}^2 = one of the eigen values of B.

Obtain a singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

Sol:- Given that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

$$B = AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$C = A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The characteristic equation of B is $|B-\lambda I| = 0$.

$$\text{i.e} \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 9-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)(-\lambda)(9-\lambda) = 0$$

$$\lambda = 0, 1, 9.$$

$\therefore$ The eigen values of B are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 9$

Let $X = [x\ y\ z]^T$ Then the matrix equation $[B-\lambda I]X = 0$.

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 9-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case(i):- An eigen vector corresponding to the eigen value $\lambda=0$:-

8?

For $\lambda=0$, the system ① can be written as .

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $x=0$, $z=0$.

choose. $y=k_1$

$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k_1 \\ 0 \end{bmatrix} = k_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the linearly independent eigen vector corresponding to eigen value $\lambda=0$.

$e_1 = \dfrac{X_1}{\|X_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the normalized eigen vector

Case(ii):- An eigen vector corresponding to the eigen value $\lambda=1$

For $\lambda=1$, the system ① can be written as .

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $y=0$, $z=0$.

Choose $x=k_2$

$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the linearly independent eigen vector corresponding to eigen value $\lambda=1$.

$e_2 = \dfrac{X_2}{\|X_2\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the normalized eigen vector .

Case (iii) :- An eigen vector corresponding to the eigen value $\lambda = 9$.

For $\lambda = 9$, The system ① can be written as.

$$\begin{bmatrix} -8 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $x = 0$, $y = 0$

choose $z = k_3$.

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 9$.

$$e_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ is the normalized eigen vector.

We observe that e_1, e_2 and e_3 are pairwise orthogonal and therefore these form an orthonormal system.

$$U = [e_1 \ e_2 \ e_3] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the matrix $C = A^TA$, the characteristic equation is $|C - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 9-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(9-\lambda)(-\lambda) = 0$$

$$\lambda = 0, 1, 9.$$

$\therefore$ The eigen values of the matrix C are $\lambda = 0, 1, 9$.

If $X = [x \ y \ z]^T$, the matrix equation $[C - \lambda I]X = 0$

$$\text{i.e} \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 9-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i) :- An eigen vector corresponding to the eigen value $\lambda = 0$.

For $\lambda = 0$, The system ② can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $x = 0$, $y = 0$

choose $z = k_1$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ is the linearly independent eigen vectors corresponding to the eigen value $\lambda = 0$.

$$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ is the normalized eigen vectors.

Case (ii) :- An eigen vector corresponding to the eigen value $\lambda = 1$.

For $\lambda = 1$, The system ② can be written as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $y = 0$, $z = 0$

choose $x = k_2$.

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$.

$$e_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is the normalized eigen vector.

Case (iii) :- An eigen vector corresponding to the eigen value $\lambda = 9$.

For $\lambda = 9$, The system ② can be written as

$$\begin{bmatrix} -8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $x = 0$, $z = 0$.

choose $y = k_3$.

$x_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k_3 \\ 0 \end{bmatrix} = k_3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the linearly independent eigen vectors corresponding to the eigen value $\lambda = 1$.

$e_3 = \frac{x_3}{\|x_3\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the normalized eigen vectors.

We observe that e_1, e_2, e_3 are pairwise orthogonal.

Therefore, these form an orthonormal system.

$$V = [e_1\ e_2\ e_3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We find that $D = U^T A V = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

$A = UDV^T$ represents the singular value decomposition of the given matrix A.

Obtain the singular value decomposition of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

Sol: Given that $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$.

$$B = AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

$$C = A^T A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

For the matrix B, the characteristic equation is $|B - \lambda I| = 0$.

i.e $\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0$.

$(11-\lambda)^2 - 1 = 0 \implies \lambda^2 - 22\lambda + 120 = 0$

$\lambda = 12, 10$.

$\therefore$ The Eigen values of the matrix B are $\lambda_1 = 12$, $\lambda_2 = 10$.

Let $X = [x \ y]^T$. Then the matrix equation $(B-\lambda I)X = 0$

i.e $\begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ——— (1)

<u>Case(i):- An eigen vector corrosponding to the eigen value $\lambda = 12$.</u>

For $\lambda = 12$, The system (1) can be written as.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this, $-x + y = 0$.

choose $y = k_1$

$x = y = k_1$

$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corrosponding to the eigen value $\lambda = 12$

$\|X_1\| = \sqrt{1+1} = \sqrt{2}$.

$e_1 = \dfrac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ is the normalized eigen vector.

<u>Case(ii):- An eigen vector corrosponding to the eigen value $\lambda = 10$</u>

For $\lambda = 10$, The system (1) can be written as

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this, $x + y = 0$

choose $x = k_2$

$y = -x = -k_2$

$X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_2 \\ -k_2 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the linearly independent eigen vector corrosponding to the eigen value. $\lambda = 10$.

$\|x_2\| = \sqrt{1+1} = \sqrt{2}$.

$e_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ is the linearly independent eigen vector normalized.

We observe that the eigen vectors e_1 and e_2 are orthogonal.

Therefore these form an orthonormal system,

$$U = [x_1 \ x_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

→ For the matrix $C = A^TA$, the characteristic equation is $|C-\lambda I| = 0$

$$\begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} = 0.$$

$$(10-\lambda)\left[(10-\lambda)(2-\lambda)-16\right] + 2\left[0-2(10-\lambda)\right] = 0$$

$$(10-\lambda)(\lambda^2 - 12\lambda) = 0.$$

$$\lambda = 12, 10, 0.$$

∴ The eigen values of the matrix are 12, 10, 0.

Let $X = [x \ y \ z]^T$ Then the matrix equation $[C-\lambda I]X = 0$.

i.e $$\begin{bmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (2)}.$$

Case (i):- An eigen vector corresponding to the eigen value $\lambda = 12$:-

For $\lambda = 12$, The system (2) can be written as

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 0 & 4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $-x+z=0$

$-y+2z=0$

Choose $z=k_1$

$x=z=k_1$

$y=2z=2k_1$

$$X_1=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}k_1\\2k_1\\k_1\end{bmatrix}=k_1\begin{bmatrix}1\\2\\1\end{bmatrix}$$ is the linearly independent eigen vector corresponding to the eigen value $\lambda=12$

$$\|X_1\|=\sqrt{1+4+1}=\sqrt{6}$$

$$e_1=\frac{X_1}{\|X_1\|}=\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$$ is the normalized eigen vector.

Case(ii): An eigen vector corresponding to the eigen value $\lambda=10$:—

For $\lambda=10$, The system (1) can be written as

$$\begin{bmatrix}0&0&2\\0&0&4\\2&4&-8\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_1\leftrightarrow R_3$

$$\begin{bmatrix}2&4&-8\\0&0&4\\0&0&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3\to 2R_3-R_2$

$$\begin{bmatrix}2&4&-8\\0&0&4\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

From this, $x+2y-4z=0$

$z=0$

$x+2y=0$

choose $y=-k_2$

$x=-2y=-2k_2$

$$X_2=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2k_2\\k_2\\0\end{bmatrix}=-k_2\begin{bmatrix}2\\-1\\0\end{bmatrix}$$ is the linearly independent eigen vector corresponding to the eigen value $\lambda=10$

$$\|X_2\|=\sqrt{4+1+0}=\sqrt{5}$$

$$e_2=\frac{X_2}{\|X_2\|}=\begin{bmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{bmatrix}$$ is the normalized eigen vector.

case(iii): An eigen vector corresponding to the eigen value $\lambda = 0$:-

For $\lambda=0$, The system ② can be written as

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 5R_3 - R_1$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 0 & 20 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this, $5x + z = 0$

$5y + 2z = 0.$

choose $x = k_3$

$z = -5x = -5k_3$

$5y = -2z = 10k_3$

$y = 2k_3$

$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_3 \\ 2k_3 \\ -5k_3 \end{bmatrix} = k_3\begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 0$.

$\|X_3\| = \sqrt{1+4+25} = \sqrt{30}$

$e_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}$ is the normalized eigen vector.

We observe that e_1, e_2, e_3 are pairwise orthogonal

Therefore they form an orthonormal system.

$$V = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

- $D = U^T A V$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 2\sqrt{2} & -\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

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$$= \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

Thus for the given matrix A, the singular value decomposition is

$A = UDV^T$.

Obtain the singular value Decomposition of the matrix $A = \begin{bmatrix} 1/\sqrt{2} & -\sqrt{3/2} \\ \sqrt{2} & 0 \end{bmatrix}$

Sol:- Given that $A = \begin{bmatrix} 1/\sqrt{2} & -\sqrt{3/2} \\ \sqrt{2} & 0 \end{bmatrix}$

$$B = AA^T = \begin{bmatrix} 1/\sqrt{2} & -\sqrt{3/2} \\ \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ -\sqrt{3/2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C = A^TA = \begin{bmatrix} 1/\sqrt{2} & \sqrt{2} \\ -\sqrt{3/2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -\sqrt{3/2} \\ \sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 5/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 3/2 \end{bmatrix}$$

The characteristic equation of B is $|B - \lambda I| = 0$ i.e $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$.

$$(2-\lambda)^2 - 1 = 0 \implies \lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda-3)(\lambda-1) = 0$$
$$\lambda = 3, 1.$$

$\therefore$ The eigen values of the matrix B are $\lambda_1 = 3$, $\lambda_2 = 1$

Let $X = [x\ y]^T$ Then the matrix equation $[B - \lambda I]X = 0$.

$$\text{i.e} \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

Case(i):- An eigen vector corresponding to the eigen value $\lambda_1 = 3$:-

For $\lambda = 3$, The system (1) can be written as

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From this, $-x+y=0$

choose $y = k_1$

$x = y = k_1$.

$$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ is an eigen vector corresponding to the eigen value $\lambda = 3$.

$$\|X_1\| = \sqrt{1+1} = \sqrt{2}.$$

$$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$ is a normalized eigen vector.

Case (ii):- An eigen vector corresponding to the eigen value $\lambda = 1$:-

For $\lambda = 1$, The system (1) can be written as

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this, $x+y=0$

choose $x = k_2$

$y = -x = -k_2$

$$X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_2 \\ -k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 1$.

$$\|X_2\| = \sqrt{1+1}$$

$$e_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$ is the normalized eigen vector.

We observe that e_1 and e_2 are orthogonal and therefore they form an orthonormal system.

$$U = [e_1 \ e_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

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$\rightarrow$ The characteristic equation of the matrix $C = A^TA$ is $|C-\lambda I| = 0$

$$\text{i.e } \begin{vmatrix} \frac{5}{2}-\lambda & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{3}{2}-\lambda \end{vmatrix} = 0 \text{ i.e } \begin{vmatrix} 5-2\lambda & -\sqrt{3} \\ -\sqrt{3} & 3-2\lambda \end{vmatrix} = 0$$

$$(5-2\lambda)(3-2\lambda) - 3 = 0$$

$$4\lambda^2 - 16\lambda + 12 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 3, 1$$

$\therefore$ The eigen values of the matrix C is are $\lambda = 3, \lambda = 1$.

Let $X = [x \;\; y]^T$ Then the matrix equation $[C-\lambda I]X = 0$.

$$\text{i.e } \begin{bmatrix} 5-2\lambda & -\sqrt{3} \\ -\sqrt{3} & 3-2\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{——— (2)}$$

Case (i):- An eigen vector corresponding to the eigen value $\lambda = 3$:-

For $\lambda = 3$, Then system (2) can be written as

$$\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \sqrt{3} R_1$$

$$\begin{bmatrix} -1 & -\sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this $\quad -x + \sqrt{3}y = 0$.

choose $y = K_1$

$x = -\sqrt{3}y = -\sqrt{3}K_1$

$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\sqrt{3}K_1 \\ K_1 \end{bmatrix} = -K_1 \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 3$.

$\|X_1\| = \sqrt{3+1} = 2$

$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ is the normalized eigen vector.

Case (ii) An eigen vector corrosponding to the eigen value $\lambda = 1$:—

For $\lambda = 1$, The system ② can be written as .

$$\begin{bmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$R_2 \longrightarrow R_2 + \frac{1}{\sqrt{3}} R_1$$

$$\begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this, $\sqrt{3}\, x - y = 0$.

choose $x = K_2$

$y = \sqrt{3}\, x = \sqrt{3} K_2$

$X_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} K_2 \\ \sqrt{3} K_2 \end{bmatrix} = K_2 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ is the linearly independent eigen vector corrospon-ding to the eigen value $\lambda = 1$.

$\|X_2\| = \sqrt{1+3} = 2$.

$e_2 = \dfrac{X_2}{\|X_2\|} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$ is the normalized eigen vector .

We observe that e_1 and e_2 are orthogonal and therefore they form an orthonormal system .

$$V = [e_1 \;\; e_2] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

We find that $D = U^T A V$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} \\ \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{3}{2}} & \frac{-1}{\sqrt{2}} \\ \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -1 \end{bmatrix}$$

With U, V and D as determined above $A = UDV^T$ gives singular value decomposition .

Sylvester's Theorem :—

This theorem is useful to find the approximate value of a matrix to a higher power and functions of matrices.

If the square matrix A has n distinct eigen values $\lambda_1, \lambda_2 \dots \lambda_n$ and $P(A)$ is a polynomial of the form.

$$P(A) = C_0 A^n + C_1 A^{n-1} + C_2 A^{n-2} + \dots + C_{n-1} A + C_n I_n .$$

Where $C_0\, C_1\, C_2 \dots C_n$ are constants then the polynomial $P(A)$ can be expressed in the following form.

$$P(A) = \sum_{r=1}^{n} P(\lambda_r) \cdot Z(\lambda_r) = P(\lambda_1) Z(\lambda_1) + P(\lambda_2) Z(\lambda_2) + \dots + P(\lambda_n) Z(\lambda_n)$$

$$\text{Where } Z(\lambda_r) = \frac{[f(\lambda_r)]}{f'(\lambda_r)}.$$

Here $f(\lambda) = |\lambda I - A|$

$[f(\lambda)]$ = Adjoint of the matrix $[\lambda I - A]$.

and $f'(\lambda_r) = \left(\frac{df}{d\lambda}\right)_{\lambda = \lambda_r}$.

(1) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, find A^{50}.

Sol:- Consider the polynomial $P(A) = A^{50}$.

Now $[\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{bmatrix}$

$$f(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{vmatrix}$$

$$f(\lambda) = (\lambda-1)(\lambda-3) = \lambda^2 - 4\lambda + 3. \quad \text{—— (1)}$$

$\therefore$ Eigen values of $f(\lambda)$ are $\lambda_1 = 1$ and $\lambda_2 = 3$.

From (1) $f'(\lambda) = 2\lambda - 4$ —— (2)

$f'(\lambda_1) = f'(1) = -2$

$f'(\lambda_2) = f'(3) = 6 - 4 = 2.$

$[f(\lambda)]$ = Adjoint matrix of the matrix $[\lambda I - A]$

$$[f(\lambda)] = \begin{bmatrix} \lambda-3 & 0 \\ 0 & \lambda-1 \end{bmatrix} \quad \text{——} (3)$$

$$z(\lambda_r) = \frac{[f(\lambda_r)]}{f'(\lambda_r)} \quad r = 1, 2., \text{ we get}$$

$$z(\lambda_1) = \frac{[f(\lambda_1)]}{f'(\lambda_1)} \qquad z(\lambda_2) = \frac{[f(\lambda_2)]}{f'(\lambda_2)}$$

$$\therefore z(\lambda_1) = z(1) = \frac{[f(1)]}{f'(1)} = -\frac{1}{2}\begin{bmatrix} 1-3 & 0 \\ 0 & 1-1 \end{bmatrix}$$

$$z(\lambda_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$z(\lambda_2) = z(3) = \frac{[f(3)]}{f'(3)} = \frac{1}{2}\begin{bmatrix} 3-3 & 0 \\ 0 & 3-1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ By Sylvester's theorem, we get

$$P(A) = P(\lambda_1)\, z(\lambda_1) + P(\lambda_2)\, z(\lambda_2)$$

$$A^{50} = \lambda_1^{50}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3^{50}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix}$$

Theorem :- The sum of the eigen values of a square matrix is equal to its trace.

proof :- We shall prove this theorem by considering a square matrix of order 3.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3 and λ be its eigen value

We prove that $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$.

The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

Expand it by using R_1, we have.

$$|A-\lambda I| = (a_{11}-\lambda)\left[(a_{22}-\lambda)(a_{33}-\lambda) - a_{32}a_{23}\right] - a_{12}\left[a_{21}(a_{33}-\lambda) - a_{31}a_{23}\right]$$
$$+ a_{13}\left[a_{21}a_{32} - a_{31}(a_{22}-\lambda)\right]$$

$$= (a_{11}-\lambda)\left[a_{22}a_{33} - a_{22}\lambda - a_{33}\lambda + \lambda^2 - a_{32}a_{23}\right] - a_{12}\left[a_{21}a_{33} - a_{21}\lambda\right.$$
$$\left.- a_{31}a_{23}\right] + a_{13}\left[a_{21}a_{32} - a_{31}a_{22} + a_{31}\lambda\right]$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{22}\lambda - a_{11}a_{33}\lambda + a_{11}\lambda^2 - a_{11}a_{23}a_{32} - a_{22}a_{33}\lambda$$
$$+ a_{22}\lambda^2 + a_{33}\lambda^2 - \lambda^3 + a_{23}a_{32}\lambda - a_{12}a_{21}a_{33} + a_{12}a_{21}\lambda - a_{31}a_{23}a_{12}$$
$$+ a_{13}a_{32}a_{21} - a_{13}a_{31}a_{22} + a_{13}a_{31}\lambda.$$

$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(a_{11}a_{22} + a_{11}a_{33} + a_{22} - a_{33} - a_{23}a_{32}$$
$$- a_{12}a_{21} - a_{13}a_{31}) + (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}) \quad \text{——} \ (1)$$

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A then.

$$|A - \lambda I| = (-1)^3(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$
$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \quad \text{——} \ (2)$$

$$|A-\lambda I| = \left[\lambda_1\lambda_2 - \lambda_1\lambda - \lambda_2\lambda + \lambda^2\right](\lambda_3-\lambda)$$

$$= \lambda_1\lambda_2\lambda_3 - \lambda\lambda_1\lambda_3 - \lambda\lambda_2\lambda_3 + \lambda^2\lambda_3 - \lambda\lambda_1\lambda_2 + \lambda^2\lambda_1 + \lambda^2\lambda_2 - \lambda^3$$

$$|A-\lambda I| = -\lambda^3 + \lambda^2(\lambda_1+\lambda_2+\lambda_3) - \lambda(\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1) + \lambda_1\lambda_2\lambda_3 \quad \text{---} (3)$$

Equating the R.H.S of (1) and (3) and comparing the coefficients of λ^2, we have

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}.$$

i.e The sum of the eigen values of A = The sum of the elements of the principal diagonal of A.

Hence The sum of the eigen values of a matrix A is equal to the trace of the matrix A.

(OR)

Another Proof :-

Let A be square matrix of order n.

The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e} \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Expanding this, we get

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) - a_{12}(\text{a polynomial of degree } n-2)$$
$$+ a_{13}(\text{a polynomial of degree } n-2) + \cdots = 0.$$

$$(-1)^n(\lambda-a_{11})(\lambda-a_{22})\cdots(\lambda-a_{nn}) + \text{a polynomial of degree } (n-2) = 0.$$

$$(-1)^n\left[\lambda^n - (a_{11}+a_{22}+a_{33}+\cdots+a_{nn})\lambda^{n-1}\right] + \text{a polynomial of degree } (n-2) \text{ in } \lambda = 0.$$

$$(-1)^n\lambda^n + (-1)^{n+1}(\text{Trace } A)\lambda^{n-1} + \text{a polynomial of degree } (n-2) \text{ in } \lambda = 0.$$

If $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the roots of this equation

$$a = (-1)^n \quad b = (-1)^{n+1} \cdot \text{Trace } A$$

$$\text{Sum of the roots} = \frac{-b}{a}$$

$$= \frac{-(-1)^{n+1} \text{Trace } A}{(-1)^n} = \text{Trace } A$$

$$\therefore \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n) = \text{Trace } A$$

Hence The sum of eigen values of a matrix A is equal to the trace of the matrix A.

<u>Theorem</u> : The Product of the eigen values of a matrix is equal to its determinant.

Proof :- Let $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ be the eigen values of square matrix A of order n.

We prove that $\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n = \det A$

The characteristic polynomial of A is

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \ldots (\lambda - \lambda_n) \text{ ——— (i)}$$

Taking $\lambda = 0$ in (i), we have

$$|A| = (-1)^n (0 - \lambda_1)(0 - \lambda_2) \ldots (0 - \lambda_n)$$

$$|A| = (-1)^n (-1)^n \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n$$

$$|A| = (-1)^{2n} \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n$$

$$|A| = \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n \quad [\because (-1)^{2n} = 1]$$

i.e The det. of A = The product of the eigen values of A.

Hence the product of the eigen values of A is equal to its determinant.

Note :- (i) If one of the eigen values of a matrix A is zero then $\det A = 0$ i.e A is singular matrix and vice versa.

(ii) If all the eigen values of a matrix A are non zero then $\det A \neq 0$ i.e A is non singular matrix and vice versa.

* - Theorem 3 :- If λ is an eigen value of A corresponding to the eigen vector X then λ^n is an eigen value of A^n corresponding to the eigen vector X.

Proof :- Given that λ is an eigen value of a matrix A and X be its corresponding eigen vector.

We P/T λ^n is an eigen value of A^n corresponding to the eigen vector X.

We prove this by using mathematical induction.

By definition, λ is an eigen value of A if there exists a non-zero vector such that $AX = \lambda X$ —— ①

The Result is true for $n=1$.

Pre multiplying eqn ① bothsides with A, weget

$$A(AX) = A(\lambda X)$$

$$A^2X = \lambda(AX)$$

$$A^2X = \lambda(\lambda X)$$

$$A^2X = \lambda^2 X \quad \text{—— ②}$$

Hence λ^2 is an eigen value of A^2 with "X" itself as the corresponding eigen vector.

Thus the theorem is true for $n=2$.

Let the result is true for $n=k$.

$$A^kX = \lambda^k X \quad \text{—— ③}$$

Pre multiplying eqn ③ bothsides with A, weget.

$$A(A^kX) = A(\lambda^k X).$$

$$A^{k+1}X = \lambda^k(AX)$$

$$A^{K+1}x = \lambda^{K+1}x.$$

Which implies that λ^{K+1} is an eigen value of A^{K+1} with x itself as the corrosponding eigen vector.

Hence by the principle of mathematical induction, the theorem is true for all positive integer n.

Hence λ is an eigen value of A corrosponding to the eigen vector x then λ^n is an eigen value of A^n corrosponding to the eigen vector x

Theorem :- A square matrix A and its transpose A^T have the same eigen values.

Proof :- Let λ be an eigen value of the matrix A.

We prove that λ is an eigen value of the matrix A^T.

We know that For any square matrix B, $|B| = |B^T|$.

$$(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I.$$

We have $|A-\lambda I| = |(A-\lambda I)^T|$

$$|A-\lambda I| = |A^T - \lambda I^T|$$

$$|A-\lambda I| = |A^T - \lambda I|$$

$\therefore |A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$.

i.e λ is an eigen value of A if and only if λ is an eigen value of A^T

Hence the Eigen values of A and A^T are same.

1) Verify that sum of the eigen values of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ is equal to its trace. and also verify that product of the eigen values of the matrix A is equal to its determinant.

Sol.- The characteristic equation of the matrix A is $|A-\lambda I| = 0$ i.e.

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

Where S_1 = Sum of the principal diagonal elements of $A = 1+2+3 = 6$.

S_2 = Sum of the minors of principal diagonal elements of A.

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-2) + (3+2) + (2-0)$$

$$S_2 = 11$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1(6-1) - 0 - 1(2-4)$$

$$S_3 = 6.$$

Hence the characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1, 2, 3.$

(i) Sum of the eigen values of A is $1+2+3 = 6$.

Trace of A is $1+2+3 = 6$.

$\therefore$ Sum of the eigen values = Trace of A.

(ii) Product of the eigen values of A is $1.2.3 = 6$.

$\det(A) = 6$.

$\therefore$ Product of eigen values = $\det(A)$.

2) Verify that an eigen values of A and A^T are same where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol:- The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

Where S_1 = sum of the principal diagonal elements of $A = 1+2+3 = 6$. 4

S_2 = Sum of the minors of principal diagonal elements of A.

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-2) + (3+2) + (2-0)$$

$$S_2 = 11$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \end{vmatrix} = 1(6-1) - 0 - 1(2-4)$$

$$S_3 = 6$$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$$\lambda = 1, 2, 3.$$

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

The characteristic equation of A^T is $|A^T - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(3-\lambda) - 2\right] - 1\left[2 - 2(2-\lambda)\right] = 0.$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 4) - (2\lambda - 2) = 0.$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0.$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

$$\lambda = 1, 2, 3.$$

We observe that eigen values of A and A^T are same.

3) Find the eigen values of the matrix A^2, where $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{vmatrix} = 0.$$

$$(2-\lambda)(-1-\lambda)-4=0.$$

$$(1+\lambda)(\lambda-2)-4=0$$

$$\lambda^2-\lambda-6=0.$$

$$\lambda=3,-2.$$

We know that λ is an eigen value of A corresponding to the eigen vector X then λ^n is an eigen value of A^n corresponding to the eigen vector X.

$\therefore$ The eigen values of A^2 are 3^2, $(-2)^2$ i.e 9, 4.

<u>Theorem</u>:- If $\lambda_1\ \lambda_2\ \lambda_3 \cdots \lambda_n$ are the eigen values of a matrix A then $k\lambda_1, k\lambda_2, \cdots k\lambda_n$ are the eigen values of the matrix KA where k is a non zero scalar.

<u>Proof</u>:- Given that $\lambda_1\ \lambda_2\ \lambda_3 \cdots \lambda_n$ are the eigen values of matrix A

We prove that $k\lambda_1, k\lambda_2, \cdots k\lambda_n$ are the eigen values of matrix KA.

Let A be a square matrix of order n.

Then $|KA-\lambda KI| = |KA-K\lambda I|$

$$= |K(A-\lambda I)|$$

$$|KA-\lambda KI| = K^n|A-\lambda I| \qquad (\because |KA| = K^n|A|)$$

Since $K \neq 0$, Therefore $|KA-\lambda KI| = 0$ iff $|A-\lambda I|=0$.

i.e $K\lambda$ is an eigen value of KA iff λ is an eigen value of A.

Thus $K\lambda_1, K\lambda_2, \cdots K\lambda_n$ are the eigen values of KA if $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ are the eigen values of A.

If $A=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$ then find the eigen values of $2A$.

Sol:- The characteristic equation of A is $|A-\lambda I|=0$

$$\text{i.e}\ \begin{vmatrix}5-\lambda & 4\\ 1 & 2-\lambda\end{vmatrix}=0$$

$$(5-\lambda)(2-\lambda)-4=0.$$

$$\lambda^2-7\lambda+6=0$$

$$\lambda=1,6.$$

We know that If λ is an eigen value of A then $k\lambda$ is an eigen value KA.

$\therefore$ The eigen values of $2A$ is 2λ i.e $2, 12$.

Theorem:- If λ is an eigen value of the matrix A then $\lambda+K$ is an eigen value of the matrix $A+KI$.

Proof:- Given that λ is an eigen value of the matrix A.

We prove that $\lambda+K$ is an eigen value of the matrix $A+KI$.

Let λ be an eigen value of A and X be the corresponding an eigen vector.

Then by the definition, $AX=\lambda X$ —— ①.

Now $(A+KI)X = AX+KIX$

$= \lambda X+KX$

$(A+KI)X = (\lambda+K)X$ ② $(\because$ From ①$)$

$\left[\because AX=\lambda X\right.$

$AX+KX=\lambda X+KX$

$\left[AX+KIX=\lambda X+KX\right.$

$\left[(A+KI)X=(\lambda+K)X\right.$

$\therefore$ By def, From ②, This show that the scalar $\lambda+K$ is an eigen value of the matrix $A+KI$ and X is an corresponding eigen vector.

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then find the eigen values of $A+30I$.

sol:- The characteristic equation of A is $|A-\lambda I| = 0$ i.e $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(2-\lambda) - 4 = 0.$$

$$\lambda^2 - 7\lambda + 6 = 0.$$

$$\lambda = 1, 6.$$

$\lambda = 1, 6$ are the eigen values of A.

We know that If λ is an eigen value of A then $\lambda+K$ is an eigen value of $A+KI$.

$\therefore$ The eigen values of the matrix $A+30I$ is $\lambda+30$

i.e 31, 36.

Theorem :- If $\lambda_1\ \lambda_2\ \lambda_3 \ldots \lambda_n$ are the eigen values of A then $\lambda_1-K, \lambda_2-K, \lambda_3-K, \ldots \lambda_n-K$ are the eigen values of the matrix $(A-KI)$ where K is a non zero scalar.

Proof :- Given that $\lambda_1\ \lambda_2\ \lambda_3 \ldots \lambda_n$ are the eigen values of A.

We prove that $\lambda_1-K, \lambda_2-K \ldots \lambda_n-K$ are the egien values of $A-KI$.

The characteristic polynomial of A is

$$|A-\lambda I| = (\lambda_1-\lambda)(\lambda_2-\lambda) \ldots (\lambda_n-\lambda) \quad \text{——— (1)}.$$

Thus the characteristic polynomial of $A-KI$ is

$$|A-KI-\lambda I| = |A-(K+\lambda)I|$$

$$= (\lambda_1-(\lambda+K))(\lambda_2-(\lambda+K)) \ldots (\lambda_n-(\lambda+K))$$

$$= ((\lambda_1-K)-\lambda)((\lambda_2-K)-\lambda) \ldots ((\lambda_n-K)-\lambda)$$

This show that the eigen values of $A-KI$ are $\lambda_1-K, \lambda_2-K \ldots \lambda_n-K$.

(OR)

Given that $\lambda_1 \lambda_2 \lambda_3 \dots \lambda_n$ are the eigen values of the matrix A.

We prove that $\lambda_1 - K, \lambda_2 - K \dots \lambda_n - K$ are the eigen values of $A - KI$.

Let λ be an eigen value of A and X be the corresponding eigen vector.

Then by the definition, $AX = \lambda X$ —— (1).

$$\begin{aligned} \text{Now } (A-KI)X &= AX - KIX \\ &= \lambda X + KX. \quad (\because \text{From (1)}) \\ (A-KI)X &= (\lambda - K)X. \text{ —— (2).} \end{aligned}$$

$$\left[\begin{aligned} &\therefore AX = \lambda X \\ &AX + KX = \lambda X + KX \\ &AX + KIX = \lambda X + KX \\ &(A+KI)X = (\lambda - K)X \end{aligned}\right.$$

$\because$ By def, From (2). This show that the scalar $\lambda - K$ is an eigen value of the matrix $A - KI$ and X is a corresponding eigen vector.

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ then find the eigen values of $A - 44I$ and $A + 2I$.

Sol:- The characteristic equation of A is $|A - \lambda I| = 0$ i.e $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0.$$

$$\lambda = 1, 6.$$

$\lambda = 1, 6$ are the eigen values of A.

We know that If λ is an eigen value of A then $\lambda - K$ is an eigen value of $A - KI$.

$\therefore$ The eigen values of the matrix $A - 44I$ is $\lambda - 44$

i.e $-43, -38$

$\therefore$ The eigen values of the matrix $A + 2I$ is $\lambda + 2$

i.e 3, 8.

Theorem :- If $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are the eigen values of A then $(\lambda_1-\lambda)^2\ (\lambda_2-\lambda)^2 \ldots (\lambda_n-\lambda)^2$ are the eigen values of $(A-\lambda I)^2$.

Proof :- Given that $\lambda_1\ \lambda_2\ \lambda_3 \ldots \lambda_n$ are the eigen values of A.

We prove that $(\lambda_1-\lambda)^2, (\lambda_2-\lambda)^2, \ldots (\lambda_n-\lambda)^2$ are the eigen values of $(A-\lambda I)^2$

First we prove that $\lambda_1-\lambda, \lambda_2-\lambda \ldots \lambda_n-\lambda$ are eigen values of $A-\lambda I$.

$\therefore$ The characteristic polynomial of A is

$$|A-kI| = (\lambda_1-k)(\lambda_2-k)\ldots(\lambda_n-k) \quad \text{—— (1).}$$

Where k is a scalar.

The characteristic polynomial of $(A-\lambda I)$ is

$$|A-\lambda I-kI| = |A-(\lambda+k)I|$$
$$= [\lambda_1-(\lambda+k)][\lambda_2-(\lambda+k)]\ldots[\lambda_n-(\lambda+k)]$$
$$= [(\lambda_1-\lambda)-k][(\lambda_2-\lambda)-k]\ldots[(\lambda_n-\lambda)-k]$$

This shows that the eigen values of $A-\lambda I$ are $\lambda_1-\lambda, \lambda_2-\lambda \ldots \lambda_n-\lambda$.

Since by the known theorem, If the eigen values of A are $\lambda_1\ \lambda_2 \ldots \lambda_n$ then the eigen values of A^n are $\lambda_1^n, \lambda_2^n, \ldots \lambda_n^n$.

Thus the eigen values of $(A-\lambda I)^2$ are $(\lambda_1-\lambda)^2\ (\lambda_2-\lambda)^2 \ldots (\lambda_n-\lambda)^2$

* Theorem :- If λ is an eigen value of a non singular matrix A corresponding to the eigen vector x then λ^{-1} is an eigen value of A^{-1} and corresponding eigen vector x itself. (OR) The eigen values of A^{-1} are the reciprocals to the eigen values of A.

Proof :- Given that A is a non singular matrix i.e $\det A \neq 0$.

We know that the product of the eigen values is equals to det A.

It follows that none of the eigen values of A is zero.

If λ is an eigen value of the non singular matrix A and x is the corresponding eigen vector then

$$AX = \lambda X \quad \text{---} (1)$$

Pre multiplying (1) by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$IX = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$\frac{1}{\lambda}X = A^{-1}X$$

$$A^{-1}X = \lambda^{-1}X$$

Hence by the definition of the eigen vector.

It follows that λ^{-1} is an eigen value of A^{-1} and X is the corresponding eigen vector.

Find the eigen values of the matrix A^{-1} where $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$.

Sol:- The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e} \begin{vmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-1-\lambda) - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$\lambda = 3, -2.$$

We know that λ is an eigen value of A corresponding to the eigen vector X.

then λ^{-1} is an eigen value of A^{-1} corresponding to the eigen vector X.

$\therefore$ The eigen values of A^{-1} are $3^{-1}, -2^{-1}$ i.e $\frac{1}{3}, \frac{-1}{2}$.

Theorem :- If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value.

Proof :- Let A be an orthogonal matrix

λ is an eigen value of A.

We prove that $\frac{1}{\lambda}$ is an eigen value of A.

Since by the known theorem, If λ is an eigen value of a non singular matrix A Then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Since A is an orthogonal matrix.

$$A^T A = A A^T = I$$

$$\therefore A^{-1} = A^T$$

$\therefore \lambda$ is an eigen value of A^T

Since by the known theorem, The square matrix A and its transpose A^T have the same eigen values.

Since determinants $|A-\lambda I|$ and $|A^T-\lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

* Theorem :- If λ is an eigen value of a non singular matrix A then $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A.

Proof :- Given that λ is an eigen value of a non singular matrix.

Therefore $\lambda \neq 0$.

λ is an eigen value of A if there exists a non zero vector X such that $AX = \lambda X$ —— (1)

Pre multiply eqn (1) by Adj A

$$(\text{Adj}A) AX = (\text{Adj}A) \lambda X$$

$$[(\text{Adj}A) A] X = \lambda (\text{Adj}A) X$$

$$|A| I X = \lambda (\text{Adj}A) X$$

$$|A| X = \lambda (\text{Adj}A) X$$

$$\frac{|A|}{\lambda} X = (\text{Adj}A) X$$

$$(\text{Adj}A) X = \frac{|A|}{\lambda} X.$$

$\therefore$ By def. It is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix AdjA

1) If eigen values of the matrix A are 2, 3 and 4. then find the eigen values of AdjA.

Sol:- Given that eigen values of A are 2, 3 and 4.

We know that If λ is an eigen value of A then $\frac{|A|}{\lambda}$ Is an eigen value of AdjA.

$$|A| = 2.3.4 = 24$$

$\therefore$ An eigen values of AdjA are $\frac{|A|}{\lambda} = \frac{24}{2} = 12,\ \frac{24}{3} = 8,\ \frac{24}{4} = 6.$

Theorem:- If A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same eigen values.

Proof:- Given that A and P be square matrices of order n.

Let $C = P^{-1}AP$

$$C - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}(AP - \lambda I P)$$

$$C - \lambda I = P^{-1}(A - \lambda I)P$$

$$|C - \lambda I| = |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}|\,|A - \lambda I|\,|P|$$

$$= |P^{-1}|\,|P|\,|A - \lambda I| \qquad [\because |P^{-1}|\,|P| = |P^{-1}P| = |I| = 1]$$

$$= |P^{-1}P|\,|A - \lambda I|$$

$$= |I|\,|A - \lambda I|$$

$$\therefore |C - \lambda I| = |A - \lambda I|$$

Thus the characteristic polynomials of C and A are same.

Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary:- If A and B are square matrices of order n and A is invertible then $A^{-1}B$ and BA^{-1} have same eigen values.

Proof:- Given that A and B are square matrices of order n.

A is invertible $\Longrightarrow A^{-1}$ is exists.

We prove that $A^{-1}B$ and BA^{-1} have same eigen values.

We know that If A and P are square matrices of order n Such that P is non singular then A and $P^{-1}AP$ have same eigen values.

Taking $A = BA^{-1}$ and $P = A$, We have.

BA^{-1} and $A^{-1}(BA^{-1})A$ have same eigen values.

BA^{-1} and $(A^{-1}B)(A^{-1}A)$ have same eigen values.

BA^{-1} and $(A^{-1}B)I$ have same eigen values.

BA^{-1} and $A^{-1}B$ have same eigen values.

Corollary:- If A and B are non singular matrices of the same order then AB and BA have the same eigen values.

Proof:- Given that A and B are non singular matrices of same order.

A is invertible $\Longrightarrow A^{-1}$ exists.

B is invertible $\Longrightarrow B^{-1}$ exists.

We have to P.T AB and BA have same eigen values.

We know that If A and P are square matrices of order n Such that P is non singular then A and $P^{-1}AP$ have same eigen values.

Taking $A = BA$ and $P = A^{-1}$, we have.

BA and $(A^{-1})^{-1}(BA)A^{-1}$ have the same eigen values

BA and $A(BA)A^{-1}$ have the same eigen values.

BA and $(AB)(AA^{-1})$ have the same eigen values.

BA and $(AB)I$ have the same eigen values.

$\therefore$ BA and AB have same eigen values

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$ then verify that AB and BA have the same eigen values.

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Sol:- Given that $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 16 \\ 4 & 2 \end{bmatrix}$$

The characteristic equation of AB is $|AB - \lambda I| = 0$

i.e $\begin{vmatrix} 14-\lambda & 16 \\ 4 & 2-\lambda \end{vmatrix} = 0$

$$(14-\lambda)(2-\lambda) - 64 = 0$$

$$\lambda^2 - 16\lambda - 36 = 0.$$

$$\lambda = 18, -2$$

$$BA = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 16 \\ 4 & 2 \end{bmatrix}$$

The characteristic equation of BA is $|BA - \lambda I| = 0$

i.e $\begin{vmatrix} 14-\lambda & 16 \\ 4 & 2-\lambda \end{vmatrix} = 0.$

$$\lambda^2 - 16\lambda - 36 = 0$$

$$\lambda = 18, -2$$

We observe that the eigen values of AB and BA are same.

Theorem:- The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof:- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & a_{nn} \end{bmatrix}_{n \times n}$ be a triangular matrix of order n.

The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ - & - & - & - & - \\ 0 & 0 & 0 & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)=0$$

$$\lambda = a_{11}, a_{22}, \cdots a_{nn}$$

Hence the eigen values of A are $a_{11}, a_{22}, a_{33} \cdots a_{nn}$

Which are just the diagonal elements of A.

<u>Note</u>:- Similarly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Eg: Find the eigen values of the matrix $A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 44 & 4 \\ 0 & 0 & 30 \end{bmatrix}$

sol:- Given that $A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 44 & 4 \\ 0 & 0 & 30 \end{bmatrix}$

The given matrix A is upper triangular matrix.

$\therefore$ The eigen values are the diagonal elements of A.

$\therefore$ The eigen values of A are 2, 44 and 30.

<u>Theorem</u> :- The eigen values of a real symmetric matrix are always real or real numbers.

<u>Proof</u>:- Let A be real symmetric matrix $\Rightarrow A^T = A$.

Let λ be an eigen value of a real symmetric matrix A and let X be the corresponding eigen vector.

Then $AX = \lambda X$ ——— ①

Take the conjugate $\bar{A}\bar{X} = \bar{\lambda}\bar{X}$

Take the transpose $(\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T$

$$\bar{X}^T\bar{A}^T = \bar{\lambda}\bar{X}^T$$

$$\bar{X}^T A^T = \bar{\lambda}\bar{X}^T \quad \text{since } \bar{A} = A$$

$$\bar{X}^T A = \bar{\lambda}\bar{X}^T \quad \text{since } A^T = A$$

Post multiply by X, we have.

$$\bar{X}^T A X = \bar{\lambda}\bar{X}^T X \text{ ——— ②}$$

Pre multiply ① by $\bar{x}^T$, we get

$$\bar{x}^T A x = \bar{x}^T \lambda x \quad \text{--- ③}$$

② − ③ gives

$(\bar{\lambda} - \lambda)\, \bar{x}^T x = 0$ [Since x is non zero vector

$\bar{\lambda} - \lambda = 0$ $\bar{x}^T$ is non zero vector

$\lambda = \bar{\lambda}$ $x\bar{x}^T \neq 0$

$\Longrightarrow \lambda$ is real

Verify that the eigen values of real symmetric matrix $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ are real.

Sol:- The characteristic equation of A is $|A - \lambda I| = 0$ i.e $\begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ -2 & 0 & 0-\lambda \end{vmatrix} = 0$

$$\text{i.e } (3-\lambda)\left[(2-\lambda)(-\lambda) - 0\right] - 2\left[0 + 2(2-\lambda)\right] = 0.$$

$$-\lambda^3 + 5\lambda^2 - 2\lambda - 8 = 0.$$

$$\lambda = -1, 2, 4.$$

We observe that the eigen values of real symmetric matrix are real.

<u>Theorem</u> :— For a real symmetric matrix, The eigen vectors corresponding to two distinct eigen values are orthogonal.

<u>Proof</u> :- Let A be a real symmetric matrix.

Let λ_1 λ_2 be eigen values of a real symmetric matrix A.

Let x_1 x_2 be the corresponding eigen vectors.

We have to prove that x_1 is orthogonal to x_2 i.e $x_1^T x_2 = 0$.

Since x_1, x_2 are eigen vectors of A corresponding to the eigen values λ_1 and λ_2

We have $Ax_1 = \lambda x_1$ —— ①

$Ax_2 = \lambda x_2$ —— ②.

Pre multiply ① by x_2^T, we get

$$x_2^T A x_1 = x_2^T \lambda_1 x_1$$

$$x_2^T A x_1 = \lambda_1 x_2^T x_1.$$

Taking transpose, we get

$$(x_2^T A x_1)^T = (\lambda_1 x_2^T x_1)^T$$

$$x_1^T A^T (x_2^T)^T = \lambda_1 x_1^T (x_2^T)^T$$

$$x_1^T A x_2 = \lambda_1 x_1^T x_2 \quad \text{—— ③}$$

Pre multiply ② by x_1^T, we get $x_1^T A x_2 = \lambda_2 x_1^T x_2$ —— ④

③ − ④, we get

$$(\lambda_1 - \lambda_2) x_1^T x_2 = 0$$

$$\Rightarrow x_1^T x_2 = 0 \quad \text{since } \lambda_1 \neq \lambda_2$$

$\therefore$ x_1 is orthogonal to x_2.

Eg: If $[1 \ 0 \ -1]^T$ $[-1 \ 2 \ -1]^T$ are eigen vectors corrosponding to two distinct eigen values of real symmetric matrix A then find the third eigen vector.

sol: Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

Let $x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the eigen vector orthogonal to x_1 and x_2.

x_1 x_3 are orthogonal $\Rightarrow$ $a + 0.b - c = 0$ —— ①.

x_2 x_3 are orthogonal $\Rightarrow$ $-a + 2b - c = 0$ —— ②.

solving ① and ②, we get

$$\begin{array}{cccc} 0 & -1 & 1 & 0 \\ 2 & -1 & -1 & 2 \end{array}$$

$$\frac{a}{2} = \frac{b}{2} = \frac{c}{2} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1}.$$

$\therefore$ $x_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the required third eigen vector.

Theorem :- The two eigen vectors corresponding to the two different eigen values are linearly independent. 11

Proof :- Let A be a square matrix.

Let x_1 and x_2 be the two eigen vectors of A corresponding to two distinct eigen values λ_1 and λ_2 Then.

$$Ax_1 = \lambda_1 x_1 \quad \text{and} \quad Ax_2 = \lambda_2 x_2 \quad \text{——} (1)$$

We prove that the eigen vectors x_1 and x_2 are L.I.

Let us assume that the eigen vectors x_1 and x_2 are L.D

By def. Then for two scalars k_1 and k_2 are not both zeros.

such that $k_1 x_1 + k_2 x_2 = 0$ —— (2)

Multiply both sides of (2) by A, we get

$$A(k_1 x_1 + k_2 x_2) = A(0) = 0.$$

$$k_1 (Ax_1) + k_2 (Ax_2) = 0.$$

$$k_1 (\lambda_1 x_1) + k_2 (\lambda_2 x_2) \text{ —— } (3) \quad (\because \text{from } (1))$$

(3) $- \lambda_2$ (2), gives

$$k_1 (\lambda_1 - \lambda_2) x_1 = 0.$$

$$k_1 = 0 \quad [\because \lambda_1 \neq \lambda_2 \text{ and } x_1 \neq 0]$$

$$\Rightarrow k_2 = 0.$$

This is contradiction to our assumption that k_1, k_2 are not zeros.

Hence our assumption x_1 and x_2 are linearly dependent is wrong

$\therefore$ x_1 and x_2 are Linearly Independent.

Theorem :— If λ is an eigen value of A then the eigen value of $B = a_0 A^2 + a_1 A + a_2 I$ is $a_0 \lambda^2 + a_1 \lambda + a_2$

Proof :- If x be an eigen vector corresponding to the eigen value λ

then $AX = \lambda X$ ——— (1).

Pre multiply by A on bothsides.

$$A(AX) = A(\lambda X)$$

$$A^2 X = \lambda (AX)$$

$$A^2 X = \lambda^2 X \quad (\because \text{from } (1))$$

By the def. This shows that λ^2 is an eigen value of A^2

We have $B = a_0 A^2 + a_1 A + a_2 I$

$$BX = (a_0 A^2 + a_1 A + a_2 I) X$$

$$= a_0 A^2 X + a_1 AX + a_2 X I.$$

$$= a_0 \lambda^2 X + a_1 \lambda X + a_2 X$$

$$BX = (a_0 \lambda^2 + a_1 \lambda + a_2) X.$$

$\therefore$ By def. This show that $a_0 \lambda^2 + a_1 \lambda + a_2$ is an eigen value of B. and the corresponding eigen vector of B is X.

Note :- If λ is an eigen value of A and $f(A)$ is any polynomial in A than the eigen value of $f(A)$ is $f(\lambda)$.

Eg:- For the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ Find the eigen values $3A^3 + 5A^2 - 6A + 2I$.

sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(3-\lambda)(-2-\lambda) = 0$$

$$\lambda = 1, 3, -2.$$

We know that if λ is an eigen value of A and $f(A)$ is a polynomial in A then the eigen value of $f(A)$ is $f(\lambda)$. 12

Let $f(A) = 3A^3 + 5A^2 - 6A + 2I$.

Eigen values of $f(A)$ are $f(1)$, $f(3)$ and $f(-2)$.

$$f(1) = 3(1)^3 + 5(1)^2 - 6(1) + 2(1) = 4$$

[$\because$ The eigen values of I are 1, 1, 1]

$$f(3) = 3 \cdot 3^3 + 5 \cdot 3^2 - 6 \cdot 3 + 2 \cdot I = 110.$$

$$f(-2) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2 \cdot 1 = 10.$$

$\therefore$ Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4, 110, 10.

Theorem :- Zero is an eigen value of a matrix iff it is singular

Proof :- Let $\lambda = 0$ is an eigen value of the matrix A

The characteristic equation of A is $|A - \lambda I| = 0$ —— (1).

$\lambda = 0$ is satisfies this equation

$$|A - 0 \cdot I| = 0$$

$$|A| = 0$$

$\implies$ A is singular

Converse :- A is singular

$\implies |A| = 0$.

$\lambda = 0$ satisfies the equation (1).

$\lambda = 0$ is an eigen value of A.

Theorem :- If X is an eigen vector of a square matrix A, then X can be not be corresponds to more than one eigen value of A.

Proof :- If possible X corresponds to two eigen values λ_1 and λ_2 of A.

then we have $AX = \lambda_1 X$ —— (1) and $AX = \lambda_2 X$ —— (2).

$$\lambda_1 X = \lambda_2 X$$

$$(\lambda_1 - \lambda_2) X = 0$$ [$\because X \neq 0$ 16c

$$\lambda_1 - \lambda_2 = 0$$ Eigen vector is must be non zero vector]

$$\lambda_1 = \lambda_2$$

Theorem :- λ is a characteristic root of a square matrix A ibb there exists a non zero vector X such that $AX = \lambda X$.

Proof :- Let λ be a characteristic root of A.

$$\therefore |A - \lambda I| = 0.$$

$$\Longrightarrow A - \lambda I \text{ is a singular matrix.}$$

$\therefore$ The homogeneous system of equations $(A-\lambda I)X = 0$ posseses non zero solution

i.e There exists a non zero vector X such that $(A-\lambda I)X = 0$.

$$AX - \lambda I X = 0$$

$$AX = \lambda X$$

Converse :- $AX = \lambda X$

$$(A - \lambda I) X = 0.$$

where X is a non zero vector.

$\therefore$ The system of homogeneous equations $(A-\lambda I)X = 0$ has a non zero solution.

Hence the coefficient matrix $A - \lambda I$ is singular

i.e $|A - \lambda I| = 0$.

This shows that λ is an eigen value of A

If 2 is an eigen value of the matrix $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ find the other two eigen values.

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Sol: Let $\lambda_1\ \lambda_2\ \lambda_3$ be the eigen values of the matrix A.

$$\lambda_1 = 2$$

Sum of the eigen values of A = sum of principal diagonal elements of A.

$$2 + \lambda_2 + \lambda_3 = 2+1-1$$

$$\lambda_2 + \lambda_3 = 0 \quad \text{—— (1)}$$

Product of the eigen values of A = Determinant of A.

$$2\lambda_2\lambda_3 = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$$2\lambda_2\lambda_3 = -8$$

$$\lambda_2\lambda_3 = -4 \quad \text{—— (2)}.$$

Solving (1) and (2), we get

$$\lambda_2 = 2 \quad \lambda_3 = -2.$$

Hence the other two eigen values are 2, -2.

If 2, 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$ find the value of a.

Sol: Let $\lambda_1\ \lambda_2\ \lambda_3$ be the eigen values of the matrix A.

$$\lambda_2 = 2 \quad \lambda_3 = 3$$

Sum of the eigen values of A = sum of principal diagonal elements of A

$$2+3+\lambda_3 = 2+2+2$$

$$\lambda_3 = 1.$$

Product of the eigen values of A = Determinant of A

$$\lambda_1\lambda_2\lambda_3 = |A|$$

$$2.3.1 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix} \Longrightarrow 6 = 8-2a.$$

$$a = 1.$$

Form the matrix whose eigen values are $\alpha-5, \beta-5, \gamma-5$ where α, β, γ are the eigen values of $A = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$

Sol: If λ_1, λ_2 and λ_3 are eigen values of the matrix A then $\lambda_1 - k, \lambda_2 - k$ and $\lambda_3 - k$ are eigen values $A - kI$.

$$\text{Required matrix} = A - 5I = \begin{bmatrix} -1 & -2 & -3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -2 & -3 \\ 4 & 0 & -6 \\ 7 & -8 & 4 \end{bmatrix}$$

Two eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and are $\frac{1}{5}$ times to the third. Find the eigen values.

Sol:- Let $\lambda_1\ \lambda_2\ \lambda_3$ be the eigen values of the matrix A.

$$\lambda_1 = \lambda_2$$

$$\lambda_1 = \frac{\lambda_3}{5}$$

$$\lambda_2 = \frac{\lambda_3}{5}.$$

Sum of the eigen values of A = Sum of principal diagonal elements of A

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$$

$$\frac{1}{5}\lambda_3 + \frac{1}{5}\lambda_3 + \lambda_3 = 7$$

$$\frac{7}{5}\lambda_3 = 7$$

$$\lambda_3 = 5$$

$$\lambda_1 = \lambda_2 = 1.$$

Hence the eigen values of A are 1, 1, 5.

Theorem :- The eigen values of a real symmetric matrix are real.

Proof :- Let A be a real symmetric matrix so that $A^T = A$

Now $\bar{A} = A$ since A is real.

$A = \bar{A}$ and $A = A^T \Rightarrow \bar{A} = A^T$

$\Rightarrow (\bar{A})^T = (A^T)^T = A$

$\Rightarrow A^\theta = A.$

$\Rightarrow A$ is Hermitian matrix.

The eigen values of a Hermitian matrix are real.

Hence the eigen values of a real symmetric matrix A are real.

Theorem :- The eigen values of a real skew symmetric matrix are all purely imaginary or zero.

Proof :- Let A be a skew symmetric matrix so that $A^T = -A$.

A is real $\Rightarrow \bar{A} = A$

$\Rightarrow (\bar{A})^T = A^T$

$\Rightarrow A^\theta = -A$

$\Rightarrow A$ is skew Hermitian matrix.

We know that the eigen values of a skew Hermitian matrix are purely imaginary or zero.

$\therefore$ It follows that the eigen values of skew symmetric matrix A are purely imaginary or zero.

Theorem :- The eigen values of an orthogonal matrix are of unit modulus.

Proof :- Let A be the orthogonal matrix so that $AA^T = I = A^TA$.

Let λ be the eigen value, x be the corresponding eigen vector of A.

So that $AX = \lambda X$ —— ①.

$$(AX)^T = (\lambda X)^T$$

$$X^T A^T = \lambda X^T \quad \text{———} \ (2)$$

Multiplying (1) and (2), we get—

$$(X^T A^T)(AX) = (\lambda X^T)(\lambda X)$$

$$X^T (A^T A) X = \lambda^2 (X^T X)$$

$$X^T I X = \lambda^2 (X^T X)$$

$$X^T X = \lambda^2 (X^T X)$$

$$(1 - \lambda^2)(X^T X) = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$|\lambda| = 1$$

$\Rightarrow$ unit modulus .

The eigen values of an ~~unitary~~ orthogonal matrix are of unit modulus .

Theorem :- The Eigen values of a hermitian matrix are real.

Proof :- Let A be a hermitian matrix i.e $A^\theta = A$ and λ be the eigen value of A.

We prove that λ is real.

If λ is an eigen value of A and x is the corresponding eigen vector then $AX = \lambda X$ —— ①.

Pre multiply both sides of ① by x^θ, we get

$$x^\theta (AX) = x^\theta (\lambda x)$$

$$x^\theta A X = x^\theta \lambda x \quad \text{—— ②}$$

Taking transposed conjugate both sides, we get.

$$(x^\theta A x)^\theta = (x^\theta \lambda x)^\theta$$

$$x^\theta A^\theta (x^\theta)^\theta = x^\theta \bar{\lambda} (x^\theta)^\theta$$

$$x^\theta A^\theta x = \bar{\lambda} x^\theta x$$

$$x^\theta A x = \bar{\lambda} x^\theta x \quad \text{—— ③} \quad [\because A^\theta = A]$$

From ② and ③, we get

$$\lambda x^\theta x = \bar{\lambda} x^\theta x$$

$$(\lambda - \bar{\lambda}) x^\theta x = 0$$

$$\lambda - \bar{\lambda} = 0$$

$$\lambda = \bar{\lambda}$$

$\therefore \lambda$ is real

$\because x$ is non zero vector
x^θ " "
$x x^\theta$ " "
i.e $x x^\theta \neq 0$

$\therefore$ Hence the eigen values of a hermitian matrix are real.

Eg. verify that the eigen values of hermitian matrix $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ are real.

sol: Given that $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$ i.e $\begin{vmatrix} 4-\lambda & 1-3i \\ 1+3i & 7-\lambda \end{vmatrix} = 0$.

$$(4-\lambda)(7-\lambda) - 10 = 0$$

$$\lambda^2 - 11\lambda + 18 = 0$$

$$\lambda = 2, 9$$

The Eigen values of A are $\lambda = 2, 9$

Which are real

$\therefore$ The Eigen values of hermitian matrix A are real.

<u>Theorem</u>:- The Eigen values of a skew hermitian matrix are either purely imaginary or zero.

<u>Proof</u>:- Let A be a skew hermitian matrix i.e $A^\theta = -A$. and λ be the eigen value of A.

We prove that $\lambda = 0$ or λ is an imaginary.

If λ is an eigen value of A and x be the corrosponding eigen vector then $AX = \lambda X$ ——— ①

Pre multiply bothsides of ① by 'i', we get

$$i(AX) = i(\lambda X)$$

$$(iA)X = (i\lambda)X.$$

By definition, $i\lambda$ is an eigen value of iA

Since A is skew hermitian, we have $A^\theta = -A$

$\Rightarrow$ iA is hermitian.

$$\text{Since } (iA)^\theta = -iA^\theta$$

$$= (-i)(-A)$$

$$(iA)^\theta = iA$$

A is skew hermitian then iA is hermitian matrix ——— ②

From ① and ②, We have $i\lambda$ is the eigen value of a hermitian matrix iA.

Since we know that Eigen values of a hermitian matrix are real. 16

$\therefore$ $i\lambda$ is real number

i.e λ is zero or purely imaginary.

Hence the Eigen values of a skew hermitian matrix are either purely imaginary or zero.

Eg: Verify that an eigen values of skew hermitian matrix $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ are either purely imaginary or zero.

Sol: Given that $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 3i-\lambda & 2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$

$$(3i-\lambda)(-i-\lambda) - (2+i)(-2+i) = 0$$

$$3 - 3i\lambda + i\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 - 2i\lambda + 8 = 0$$

$$\lambda = \frac{2i \pm \sqrt{-4-32}}{2} = \frac{2i \pm 6i}{2} = i \pm 3i$$

$$\lambda = 4i, -2i$$

The Eigen values of A are $\lambda = 4i, -2i$

Which are purely imaginary.

$\therefore$ The Eigen values of given skew hermitian matrix are purely imaginary.

<u>Theorem</u>:- The Eigen values of unitary matrix is of unit modulus.

<u>Proof</u>:- Let A be a unitary matrix i.e $AA^{\theta} = I = A^{\theta}A$ and λ be the Eigen value of A

We prove that $|\lambda| = 1$.

If λ is an eigen value of A and x be the corresponding eigen vector then $AX = \lambda X$ ——— ①.

Taking transposed conjugate on bothsides of ①, we get

$$(AX)^\theta = (\lambda X)^\theta$$

$$X^\theta A^\theta = \bar{\lambda} X^\theta \quad \text{---} ②$$

Multiplying ① and ②, we get

$$(X^\theta A^\theta)(AX) = (\bar{\lambda} X^\theta)(\lambda X)$$

$$X^\theta (A^\theta A) X = \lambda\bar{\lambda}(X^\theta X)$$

$$X^\theta I X = \lambda\bar{\lambda}(X^\theta X) \qquad \left[\because AA^\theta = I = A^\theta A,\ A \text{ is unitary}\right.$$

$$X^\theta X = \lambda\bar{\lambda}(X^\theta X)$$

$$(1-\lambda\bar{\lambda}) X^\theta X = 0$$

$$1-\lambda\bar{\lambda} = 0 \qquad \left[\because X \text{ is non zero vector}\right.$$

$$\lambda\bar{\lambda} = 1 \qquad X^\theta \text{ is non zero vector}$$

$$|\lambda|^2 = 1 \qquad XX^\theta$$

$$|\lambda| = 1 \qquad \text{i.e } XX^\theta \neq 0$$

Hence the Eigen values of a unitary matrix are of unit modulus.

Eg Verify that the eigen values of a unitary matrix $A=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ are of unit modulus

sol: Given that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e} \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} - \lambda \end{vmatrix} = 0$$

$$-\left(\frac{1}{\sqrt{2}}+\lambda\right)\left(\frac{1}{\sqrt{2}}-\lambda\right) - \frac{1}{2} = 0.$$

$$-\left(\frac{1}{2}-\lambda^2\right) - \frac{1}{2} = 0.$$

$$\lambda^2 - 1 = 0$$

$\lambda = \pm 1$, Eigen values of A are $\lambda = 1, -1$

$$|\lambda| = 1$$

$\therefore$ Eigen values of unitary matrix A are of unit modulus.

Theorem :- An Eigen values of Idempotent matrix are 0 and 1

Proof :- Let A be an Idemponent matrix i.e $A^2 = A$. —— ①

Let λ be an eigen value of A and X is corresponding eigen vector. Then

$$AX = \lambda X \text{ —— ②}$$

We prove that an eigen values of A are 0 and 1 i.e $\lambda = 0$ and 1.

We know that If λ is an eigen value of A corresponding to the eigen vector X then λ^n is an eigen value of A^n corresponding to the eigen vector X.

We have $A^n X = \lambda^n X$

$$\Rightarrow A^2 X = \lambda^2 X \text{ —— ③}$$

From ② and ③, we get

$$AX = \lambda^2 X \text{ —— ④}$$

From ② and ④, we get

$$\lambda^2 X = \lambda X$$

$$(\lambda^2 - \lambda) X = 0.$$

$$\lambda^2 - \lambda = 0 \qquad [\because X \neq 0]$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0, \lambda = 1$$

$\therefore$ An Eigen values of Idempotent matrix A are 0 and 1.

QUADRATIC FORMS

Quadratic form :—

A homogeneous polynomial of second degree in n variables $x_1, x_2, x_3 \ldots x_n$ is called a quadratic form in the n variables.

It is denoted by Q.

Thus $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ is a quadratic form in n variables $x_1, x_2, \ldots x_n$

[OR]

An expression of the form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ where a_{ij}'s are elements of a field F is called a quadratic form in n variables. $x_1, x_2, x_3 \ldots x_n$ over a field F.

If a_{ij}'s belongs to a real number field R then the above quadratic form is said to be a "real quadratic form" in n variables $x_1, x_2, \ldots x_n$

It is denoted by Q i.e $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$

Eg:- (i) $Q = x^2$ is a quadratic form in a single variable x.

(ii) $Q = 3x^2 + 4xy + 7y^2$ is a quadratic form in two variables x, y.

(iii) $Q = x^2 + y^2 + 3z^2 + 4xy - 7xz + 8yz$ is a quadratic form in 3 variables.

Quadratic form corresponding to a Real Symmetric Matrix :—

Let $A = [a_{ij}]_{n\times n}$ be a real symmetric matrix and let $X = [x_1\ x_2\ x_3 \ldots x_n]^T$ be a column matrix Then $X^T A X$ will determine a quadratic form

$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ ..

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j = a_{11} x_1^2 + a_{12} x_1 x_2 + \ldots + a_{1n} x_1 x_n + a_{21} x_2 x_1 + a_{22} x_2^2 + \ldots + a_{2n} x_2 x_n$$
$$+ \ldots + a_{n1} x_n x_1 + a_{n2} x_n x_2 + \ldots + a_{nn} x_n^2$$

$$= a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + \ldots + (a_{1n} + a_{n1}) x_1 x_n + a_{22} x_2^2 +$$
$$(a_{23} + a_{32}) x_2 x_3 + \ldots + (a_{2n} + a_{n2}) x_2 x_n + \ldots + a_{nn} x_n^2$$

Matrix of a Quadratic form :-

If $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ is a quadratic form in n variables $x_1, x_2, \dots x_n$ over a field F then there exists a unique symmetric matrix A of order n such that $Q = X^T A X$

Where $X = [x_1\ x_2\ x_3\ \dots\ x_n]^T$

Here the symmetric matrix A is called the matrix of the quadratic form Q.

Find the quadratic form relating to the symmetric matrix $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 4 \end{bmatrix}$

sol: Let $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 4 \end{bmatrix}$

The Quadratic form related to the given matrix is $X^T A X$.

Where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $X^T = [x_1\ x_2\ x_3]$.

$\therefore$ Required quadratic form $= X^T A X = [x_1\ x_2\ x_3] \begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= [x_1\ x_2\ x_3] \begin{bmatrix} x_1 + 2x_2 + 6x_3 \\ 2x_1 + x_2 + 3x_3 \\ 6x_1 + 3x_2 + 4x_3 \end{bmatrix}$$

$$= x_1(x_1 + 2x_2 + 6x_3) + x_2(2x_1 + x_2 + 3x_3) + x_3(6x_1 + 3x_2 + 4x_3)$$

$$= x_1^2 + x_2^2 + 4x_3^2 + 4x_1x_2 + 6x_2x_3 + 12x_1x_3.$$

Write down the ~~qua~~ symmetric matrix. of the quadratic form.

$2x_1^2 + 3x_2^2 + 44x_3^2 - 3x_1x_2 + 4x_2x_3 - 5x_1x_3$.

sol:- Given that $2x_1^2 + 3x_2^2 + 44x_3^2 - 3x_1x_2 + 4x_2x_3 - 5x_1x_3$.

It can be written as $2x_1^2 + 3x_2^2 + 44x_3^2 - \frac{3}{2}x_1x_2 - \frac{3}{2}x_2x_1 + 2x_2x_3 + 2x_3x_2 - \frac{5}{2}x_1x_3 - \frac{5}{2}x_3x_1$

$\therefore$ The Matrix of quadratic form $A = \begin{bmatrix} 2 & -3/2 & -5/2 \\ -3/2 & 3 & 2 \\ -5/2 & 2 & 44 \end{bmatrix}$

Linear Transformation of a Quadratic form :-

1

Let $Q = X^T A X$ be a quadratic form in n variables $x_1, x_2, x_3 \dots x_n$ and the symmetric matrix $A = [a_{ij}]_{n \times n}$ be the matrix of Q.

Let $X = PY$ be a non singular transformation when P is a non singular matrix of order n and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$

$$X = PY$$

$$X^T = (PY)^T = Y^T P^T$$

$$Q = X^T A X$$

$$= (PY)^T A (PY)$$

$$= Y^T (P^T A P) Y$$

$$Q = Y^T B Y \qquad \text{Where } B = P^T A P$$

$$B^T = (P^T A P)^T$$

$$= P^T A^T (P^T)^T$$

$$= P^T A P$$

$$B^T = B$$

$\therefore$ B is symmetric.

Hence $Y^T B Y$ is another quadratic form in n variables $y_1, y_2, y_3 \dots y_n$

Thus the linear transformation $X = PY$ transforms the given quadratic form Q to another quadratic form $Q' = Y^T B Y$.

i.e $Y^T B Y$ is the linear transform of $X^T A X$ under the linear transform $X = PY$.

If P is a non singular matrix of order n, then the linear transformation $X = PY$ is said to be a non singular linear transformation

A non singular transformation is also called regular transformation

If P is an orthogonal matrix of order n then the linear trans-formation $X = PY$ is called an orthogonal transformation.

2

Canonical form or Normal form of a quadratic form :-

A real quadratic form in which the product terms are missing and which contains only terms of squares of variables is called a canonical form.

Eg:- $Q = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + \cdots + a_n x_n^2$ is a canonical form.

[OR]

If $X^T A X$ is a real quadratic form in n variables, then there exists a real non singular linear transformation $X = PY$ which transforms $X^T A X$ to the form $y_1^2 + y_2^2 + y_3^2 + \cdots + y_r^2$.

This expression is called the canonical form or normal form of the given quadratic form $X^T A X$.

Rank of a Quadratic form :-

Let $X^T A X$ be a quadratic form over a field F. The rank of the matrix A is called the rank of the quadratic form $X^T A X$.

Working procedure for the reduction of Quadratic form to the Normal form or Canonical form :-

Let $Q = X^T A X$ be a quadratic form of n-variables.

Let A be the matrix of the ~~matrix~~ quadratic form.

Here A is the symmetric matrix.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Step(i) :- We can write $A_{3\times3} = I_3 A I_3$ —— (1)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now reduce the matrix A on the L.H.S to the diagonal form by applying a finite no. of elementary transformations. Each row transformation will be applied to the pre factor I_3 and each column transformation applied to the post factor I_3 on the R.H.S of eqn(1)

Step(ii):- If $a_{11} \neq 0$ then by using a_{11} position make a_{21}, a_{31} positions as zero. The same row operations will be applied pre factor of A on R.H.S.

3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} \times & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step(iii):- By using a_{11} position make a_{12}, a_{13} positions as zero. The same column operations will be applied post factor of A on R.H.S.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & a''_{23} \\ 0 & a''_{32} & a''_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} \times & & \\ & & \\ & & \end{bmatrix}$$

Step(iv):- If $a''_{22} \neq 0$ then by using a''_{22} position make a''_{32} position as zero. The same row operation will be applied prefactor of A on R.H.S

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & a''_{23} \\ 0 & 0 & a'''_{33} \end{bmatrix} = \begin{bmatrix} \times & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Step(v):- By using a''_{22} position make a''_{23} position as zero. The same column operations will be applied post factor of A on R.H.S

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a''_{22} & 0 \\ 0 & 0 & a^{iv}_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} A \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

The Resulting equation is P^TAP = Diagonal matrix.

Where P is a non singular matrix of order n.

Step (vi) :- Finally we can interpret the above result interms of quadratic forms.

(4)

If $X^T A X$ be a real quadratic form in n variables then there exists a linear transformation $X = PY$ where P is a non singular matrix of order n, transforms the quadratic form $X^T A X$ to a diagonal form.

i.e $Y^T P^T A P Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \dots + \lambda_r y_r^2$

i.e a sum of r - square terms. Here r gives the rank of the quadratic form $X^T A X$.

Note :- In the above procedure of diagonal form if we make the diagonal elements as 1 or -1 or 0 then we obtain the required canonical form or normal form of the given quadratic form.

Index of the quadratic form :-

Let $y_1^2 + y_2^2 + y_3^2 + \dots + y_p^2 - y_{p+1}^2 + \dots + y_r^2$ be a canonical form or normal form of real quadratic form $X^T A X$. The number of positive terms in the normal form of $X^T A X$ is called the index of the quadratic form.

It is denoted by s.

The numbers of non positive terms is equal to $r - s$.

Signature of the quadratic form :-

The difference of the numbers of positive terms and the non-positive terms is called the signature of the quadratic form.

$\therefore$ Signature $= s - (r - s) = 2s - r$.

Nature of a Quadratic form :-

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The quadratic form $X^T A X$ in n variables is said to be

(i) Positive Definite :- If $r = n$ and $s = n$ (OR) If all the eigen values of A are positive.

(ii) Nagative Definite :- If $r = n$ and $s = 0$ (OR) If all the eigen values of A are $-ve$.

(iii) Positive semi definite :- If $r < n$ and $s = r$ [OR] If all the eigen values of A are ≥ 0 and atleast one eigen value is zero.

(iv) Nagative semi definite :- If $r < n$ and $s = 0$ [OR] If all the eigen values of A ≤ 0 and atleast one eigen value is zero.

(v) Indefinite :- In all other cases [OR] If A has positive as well as negative eigen values.

1) Identify Nature, Index, Rank and signature of the quadratic form $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$.

Sol:- The given quadratic form can be written as $x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_1 + x_1x_3 + x_3x_1 - 2x_2x_3 - 2x_3x_2$

The matrix of the quadratic form is $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{i.e } \begin{vmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$R_1 \to R_1 + R_2 + R_3$$

$$\begin{vmatrix} -\lambda & -\lambda & -\lambda \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} 1 & 1 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$C_2 \to C_2 - C_1, \quad C_3 \to C_3 - C_1$$

$$-\lambda \begin{vmatrix} 1 & 0 & 0 \\ -2 & 6-\lambda & 0 \\ 1 & -3 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \left[-\lambda(6-\lambda) - 0\right] = 0$$

$$\lambda = 0, 0, 6.$$

$\therefore$ The Eigen values of A are $\lambda = 0, 0, 6$.

(i) The Nature of the quadratic form is positive semi definite.

(ii) The Index of the quadratic form is 1.

(iii) Rank of the quadratic form is 1.

(iv) Signature of the ~~to~~ quadratic form is $2s - r = 1$.

6

Find the transformation which will transform $4x^2+3y^2+z^2-8xy-6yz+4zx$ into a sum of squares and find the reduced form.

sol:- Given that the quadratic form

$$4x^2+3y^2+z^2-8xy-6yz+4zx.$$

It can be written as

$$4x^2+3y^2+z^2-4xy-4yx-3yz-3zy+2zx+2xz.$$

The matrix of the quadratic form is

$$A=\begin{bmatrix}4 & -4 & 2\\ -4 & 3 & -3\\ 2 & -3 & 1\end{bmatrix}$$

We write $A=I_3AI_3$.

We apply elementary operations on A of L.H.S and we apply the same row operations on the pre factor and column operations on the post factor

$$\begin{bmatrix}4 & -4 & 2\\ -4 & 3 & -3\\ 2 & -3 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$R_2 \to R_2+R_1 \quad R_3 \to 2R_3+R_1$

$$\begin{bmatrix}4 & -4 & 2\\ 0 & -1 & -1\\ 0 & -2 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ -1 & 0 & 2\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$C_2 \to C_2+C_1 \quad C_3 \to 2C_3-C_1$

$$\begin{bmatrix}4 & 0 & 0\\ 0 & -1 & -2\\ 0 & -2 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ -1 & 0 & 2\end{bmatrix}A\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$C_2 \longrightarrow 3C_2 + C_1 \qquad C_3 \longrightarrow 3C_3 - C_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & -3 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$R_3 \longrightarrow 7R_3 + R_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & 0 & 144 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$C_3 \longrightarrow 7C_3 + C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & -21 \end{bmatrix}$$

This is of the form $B = P^T A P$.

$B = \text{Diag}\{6, 21, 1008\} = P^T A P$.

Thus the non singular linear transformation $X = PY$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & -21 \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ transforms the given quadratic form to to the diagonal form which is given by $6y_1^2 + 21y_2^2 + 1008y_3^2$.

Rank of the quadratic form $r = 3$.

Index of the quadratic form $s = 3$

Signature of the quadratic form $2s - r = 6 - 3 = 3$.

$\therefore$ The given quadratic form is positive definite.

$\therefore$ The required nonsingular linear transformation which brings about an diagonal form is $X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & -21 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x = y_1 + y_2 - 6y_3 \qquad y = 3y_2 + 3y_3, \quad z = -21y_3$

QUADRATIC FORMS

I: Reduce the following quadratic forms into a sum of squares. Indicate the nature, rank, index and signature of the quadratic form. Also write the corresponding linear transformation which brings about the normal reduction.

R.NO	Q.NO
1-20	1-3
21-40	4-6
41-60	7-10

(i) $3x^2+3y^2+3z^2+4xy+8xz+8yz$.

Ans:- Rank $=3$, Index $=2$, Nature: Indefinite.

(ii) $4x^2+3y^2+z^2-8xy-6yz+4zx$.

Ans:- Rank $=3$, Index $=2$, Nature: Indefinite.

(iii) $5x^2+26y^2+10z^2+4yz+14zx+6xy$.

Ans:- Rank $=2$, Index $=2$ Nature: Positive semi definite.

(iv) $7x^2+6y^2+5z^2-4xy-4yz$

Ans:- Rank $=3$, Index $=3$, Nature: Positive definite.

(v) $3x^2+2y^2+z^2+4xy-2xz+6yz$

Ans:- Rank $=3$, Index $=2$ (Nature: Indefinite.

(vi) $-3x^2-3y^2-3z^2-2xy-2yz+2xz$

Ans:- Rank $=$ Index $=$ Nature: Nagative definite.

(vii) $6x^2+17y^2+3z^2-20xy-14yz+8zx$

Ans:- Rank $=2$ Index $=2$ Nature: Positive semi definite.

(viii) $6x^2+3y^2+14z^2+4yz+18xz+4xy$.

Ans:- Rank $=3$ Index $=3$ Nature: Positive definite.

(ix) $4x_1^2+9x_2^2+2x_3^2+8x_2x_3+6x_3x_1+6x_1x_2$

Ans:- Rank $=3$ Index $=2$ Nature Indefinite.

(x) $6x_1^2+3x_2^2+3x_3^2-4x_1x_2+4x_1x_3-2x_2x_3$.

Ans:- Rank $=3$ Index $=3$ Nature: Positive definite.

II. Reduce the following quadratic forms to canonical form. In each case find the matrix of the transform. Also find rank, index, nature and signature of the quadratic form.

R.NO	Q.NO
1–20	1–3
21–40	4–6
41–60	7–10

(i) $2xy + 2yz + 2zx$.

Ans:- Rank = 3, Index = 1 Nature: Indefinite.

(ii) $2x^2 + 2y^2 + 2z^2 - 2xy + 2xz - 2yz$

Ans:- Rank = 3 Index = 3 Nature Positive definite.

(iii) $3x^2 + 3z^2 + 4xy + 8xz + 8yz$

Ans:- Rank = 3, Index = 1 Nature: Indefinite.

(iv) $x^2 + 4y^2 + z^2 + 4xy + 6yz + 2zx$

Ans:- Rank = Index = Nature:

(v) $x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt$.

Ans:- Rank = 3 Index = 2 Nature: Indefinite.

(vi) $2x^2 + y^2 - 3z^2 + 12xy - 4zx - 8yz$.

Ans:- Rank = 3 Index = 1. Nature: Indefinite.

(vii) $x_1^2 + 3x_2^2 + 5x_3^2 - 4x_1x_2 + 2x_3x_1 + 4x_2x_3$.

Ans:- Rank = 3 Index = 2 Nature: Indefinite.

(viii) $2xy - 4yz - 6zx$

Ans:- Rank = Index = Nature:

(ix) $9x^2 + 2y^2 + 2z^2 + 6xy + 2yz - 2zx$

Ans:- Rank = 3 Index = 3 Nature: Positive definite.

(x) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$.

Ans:- Rank = 3 Index = 3 Nature: Positive definite.

Reduction of the Quadratic form to canonical form by Orthogonal Transformation :-

If in the transformation $X = PY$, P is an orthogonal matrix and if $X = PY$ transforms the Quadratic form Q to the canonical form the Q is said to be reduced to the canonical form by an orthogonal transformation.

Working procedure :-

Let $Q = X^T A X$ be a given quadratic form.

Step1 :- Let A be the matrix of the quadratic form.

Step2 :- The characteristic equation of A is $|A - \lambda I| = 0$

Solve the characteristic equation and find the eigen values $\lambda_1, \lambda_2, \lambda_3$ of the matrix A.

Step 3 :-

case(i) :- If the eigen values $\lambda_1, \lambda_2, \lambda_3$ of the matrix A are distinct.

Step(i) :- Find the eigen vectors X_1, X_2, X_3 corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$ and these eigen vectors are linearly independent. observe that these eigen vectors are pairwise orthogonal.

$\therefore$ The matrix A is diagonalizable.

Step(ii) :-

$$\text{Modal Matrix} = [X_1 \ X_2 \ X_3] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Step(iii) :- Construct the normalized eigen vectors e_1, e_2, e_3 corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$.

Where $e_1 = \dfrac{X_1}{\|X_1\|}$ $\quad e_2 = \dfrac{X_2}{\|X_2\|}$ $\quad e_3 = \dfrac{X_3}{\|X_3\|}$

$$\|X_1\| = \sqrt{a_1^2 + b_1^2 + c_1^2} \quad \|X_2\| = \sqrt{a_2^2 + b_2^2 + c_2^2} \quad \|X_3\| = \sqrt{a_3^2 + b_3^2 + c_3^2}$$

Step (iv):- Define the normalized modal matrix.

$$P = [e_1\ e_2\ e_3] = \left[\frac{X_1}{\|X_1\|}\ \frac{X_2}{\|X_2\|}\ \frac{X_3}{\|X_3\|}\right]$$

2

This P will be an orthogonal matrix.

By definition of an orthogonal matrix.

$$PP^T = P^TP = I$$

$$\Longrightarrow P^{-1} = P^T$$

Step (v):- Find $P^{-1}AP$ (OR) P^TAP

Which is the diagonal matrix of A.

$$P^{-1}AP = P^TAP = D = \text{Diag}[\lambda_1\ \lambda_2\ \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Step (vi):- Now Define the orthogonal transformation $X = PY$

Which transforms the given Quadratic form $Q = X^TAX$ to the normal form is given by $Q = X^TAX$ Where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$Q = (PY)^T A (PY)$$

$$= (Y^TP^T) A (PY)$$

$$= Y^T (P^TAP) Y$$

$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$Q = Y^TDY$$

$$Q = [y_1\ y_2\ y_3]^T \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

Which is the required Normal form (OR) Canonical form.

Case (ii):- If the eigen values $\lambda_1\ \lambda_2\ \lambda_3$ of the matrix A are not distinct. If suppose λ_1 is repeated two times.

Step (i):- Find the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$ and these eigen vectors are linearly independent.

If Algebraic multiplicity of an eigen value $\lambda \neq$ Geometric multiplicity of an eigen value λ then Diagonalization for the matrix A is not possible. (3)

So we stop the procedure.

else (Algebraic multiplicity of an eigen value λ = Geometric multiplicity of an eigen value λ)

$\therefore$ The matrix A is diagonalizable.

goto step(ii).

<u>step(ii)</u> :- Here we observe that the eigen vectors X_1, X_2 are not pairwise orthogonal corresponding to the eigen value λ_1.

Now we find the eigen vector X_1 is pairwise orthogonal to X_2 and X_3

Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ is pairwise orthogonal to X_2 and X_3.

Where $X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$ $X_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$

X_1, X_2 are pairwise orthogonal if $x_1 a_2 + y_1 b_2 + z_1 c_2 = 0$

X_1, X_3 are pairwise orthogonal if $x_1 a_3 + y_1 b_3 + z_1 c_3 = 0$.

Solve the above equations, we get the values of x_1, y_1 and z_1

$\therefore$ The Eigen vectors X_1, X_2 and X_3 are pairwise orthogonal.

<u>step(iii)</u> :- Modal Matrix $= [X_1 \ X_2 \ X_3] = \begin{bmatrix} x_1 & a_2 & a_3 \\ y_1 & b_2 & b_3 \\ z_1 & c_2 & c_3 \end{bmatrix}$

step(iv) :- Construct the normalized eigen vectors e_1, e_2, e_3 corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$.

$\|X_1\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$ $\|X_2\| = \sqrt{a_2^2 + b_2^2 + c_2^2}$ $\|X_3\| = \sqrt{a_3^2 + b_3^2 + c_3^2}$

Where $e_1 = \dfrac{X_1}{\|X_1\|}$ $e_2 = \dfrac{X_2}{\|X_2\|}$ $e_3 = \dfrac{X_3}{\|X_3\|}$

Step (v) :- Normalized Modal Matrix

$$P = [e_1 \ e_2 \ e_3] = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix}$$

This P will be an orthogonal matrix.

By definition of an orthogonal matrix $PP^T = P^TP = I \Rightarrow P^{-1} = P^T$

Step (vi) :- Find $P^{-1}AP$ (or) P^TAP

which is the diagonal matrix of A.

$$P^{-1}AP = P^TAP = D = \text{Diag}[\lambda_1 \ \lambda_2 \ \lambda_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Now define an orthogonal transformation $X = PY$ which transforms the given Quadratic form $Q = X^TAX$ to the normal form is given by.

$$Q = X^TAX \qquad \text{where } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

$$= (PY)^T A (PY)$$

$$= (Y^T P^T) A (PY)$$

$$= Y^T (P^TAP) Y$$

$$Q = Y^T D Y$$

$$= [y_1 \ y_2 \ y_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

Which is the required normal form or canonical form.

Pairwise Orthogonal :—

Let A be a square matrix (Symmetric) of order 3.

If $\lambda_1 \ \lambda_2 \ \lambda_3$ are three distinct eigen values of A then the corresponding eigen vectors X_1, X_2 and X_3 are pairwise orthogonal.

Let $X_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \quad X_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}$

X_1, X_2 are pairwise orthogonal if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

X_2, X_3 are pairwise orthogonal if $a_2a_3 + b_2b_3 + c_2c_3 = 0$

X_1, X_3 are pairwise orthogonal if $a_1a_3 + b_1b_3 + c_1c_3 = 0$

(1) Reduce the quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ to the normal form by orthogonal transformation.

Sol:- Given that $Q = 3x^2+2y^2+3z^2-2xy-2yz$.

The above quadratic form can be written as

$$Q = 3x^2+2y^2+3z^2-xy-yx-yz-zy$$

The matrix of the quadratic form is

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is $|A-\lambda I| = 0$

i.e $\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$

$$(3-\lambda)\left[(2-\lambda)(3-\lambda)-1\right]+1\left[(-1)(3-\lambda)\right] = 0$$

$$(3-\lambda)\left[6-3\lambda-2\lambda+\lambda^2-1-1\right] = 0$$

$$(3-\lambda)(\lambda^2-5\lambda+4) = 0$$

$$(3-\lambda)(\lambda-4)(\lambda-1) = 0$$

$$\lambda = 1, 3, 4.$$

The eigen values of A are $\lambda = 1, 3, 4$.

These eigen values are distinct.

$\therefore$ The matrix A is diagonalizable.

Now the Eigen vector corresponding to the Eigen values λ are obtained by solving the system of equations $(A-\lambda I)X = 0$.

i.e $\begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case(ii): - Eigen vector corresponding to the Eigen value $\lambda = 3$: —

For $\lambda = 3$, The system ① can be written as

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coeff. matrix into echelon form by applying E-row operations only

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Rank of the coefficient matrix is 2 i.e $\rho(A) = 2$ = The No. of non zero rows equivalent to A.

So that the homogeneous system has $n - r = 3 - 2 = 1$ L.I solution.

There is only one linearly independent eigen vector corresponding to the eigen value $\lambda = 3$.

To determine this, we have to assign an arbitrary value for one variable.

From the above system the linear equations are

$$x + y + z = 0$$

$$y = 0.$$

$$x + z = 0$$

choose $x = k_1$

$z = -x = -k_1$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ -k_1 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is the eigen vector corrosponding to the eigen value $\lambda = 3$.

Case (ii) :- Eigen vector corrosponding to the eigen value $\lambda = 1$:-

For $\lambda = 1$, The system ① can be written as

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coeff. matrix into echelon form by applying E-row operations only

$R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + 2R_1$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + R_2$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \rho(A) = 2 =$ The No. of Non zero rows equivalent to A

So that the homogeneous system has $n - r = 3 - 2 = 1$ L.I solution.

There is only one linearly independent eigen vector corrosponding to the eigen value $\lambda = 1$.

To determine this, we have to assign an arbitrary value for one variable.

From the above system the linear equations are

$$-x + y - z = 0$$

$$y - 2z = 0.$$

$\therefore$ This matrix P will reduce the matrix A to be diagonal form which is given by $P^{-1}AP = D$

$$\text{i.e } P^TAP = D$$

$$D = P^TAP = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus the orthogonal transformation $X = PY$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ transforms the given quadratic form to the normal form is given by.

$$Q = X^TAX$$

$$Q = (PY)^TA(PY)$$

$$Q = Y^T(P^TAP)Y$$

$$Q = Y^TDY$$

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = 3y_1^2 + y_2^2 + 4y_3^2$$

$\therefore$ The required orthogonal transformation which brings about the normal form is given by $X = PY$ i.e $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$x = \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$, $y = \frac{2}{\sqrt{6}}y_2 - \frac{1}{\sqrt{3}}y_3$ $z = -\frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{3}}y_3$

The Rank of the Q.F $r = 3$, Index of the Q.F $= s = 3$.

signature of the Q.F $= 2s - r = 3$ Nature of Q.F is +ve definite.

Reduce the quadratic form $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$ into sum of squares form by an orthogonal transformation and give the matrix of transformation.

sol:- Given that $3x_1^2+3x_2^2+3x_3^2+2x_1x_2+2x_1x_3-2x_2x_3$

The given Q.F can be written as

$$3x_1^2+3x_2^2+3x_3^2+x_1x_2+x_2x_1+x_1x_3+x_3x_1-x_2x_3-x_3x_2$$

The matrix of the quadratic form is

$$A=\begin{bmatrix}3&1&1\\1&3&-1\\1&-1&3\end{bmatrix}$$

The characteristic equation of A is $|A-\lambda I|=0$

$$\text{i.e}\begin{vmatrix}3-\lambda&1&1\\1&3-\lambda&-1\\1&-1&3-\lambda\end{vmatrix}=0$$

$$R_1\longrightarrow R_1+R_3$$

$$\begin{vmatrix}4-\lambda&0&4-\lambda\\1&3-\lambda&-1\\1&-1&3-\lambda\end{vmatrix}=0$$

$$(4-\lambda)\begin{vmatrix}1&0&1\\1&3-\lambda&-1\\1&-1&3-\lambda\end{vmatrix}=0.$$

$$R_3\longrightarrow R_3-R_2$$

$$(4-\lambda)\begin{vmatrix}1&0&1\\1&3-\lambda&-1\\0&\lambda-4&4-\lambda\end{vmatrix}=0.$$

$$(4-\lambda)^2\begin{vmatrix}1&0&1\\1&3-\lambda&-1\\0&-1&1\end{vmatrix}=0.$$

$C_3 \longrightarrow C_3 - C_1$

$$(4-\lambda)^2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 3-\lambda & -2 \\ 0 & -1 & 1 \end{vmatrix} = 0$$

$$(4-\lambda)^2 \left[(3-\lambda) - 2\right] = 0 .$$

$$(4-\lambda)^2 \cdot (\lambda+1) = 0$$

$$\lambda = 1, 4, 4 .$$

The eigen values of A are $\lambda = 4, 4, 1$.

The algebraic multiplicities of an eigen values 4 and 1 are 2 and 1.

Now we have to find the eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ corrosponding to

the eigen values λ are obtained by solving the system of equations

$(A - \lambda I) X = 0$ i.e. $\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ —— (1)

Case(i):- Eigen vector corrosponding to the eigen value $\lambda = 1$:—

For $\lambda = 1$, The system (1) can be written as

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and determine the rank of the matrix

$$R_2 \longrightarrow 2R_2 - R_1 \quad R_3 \longrightarrow 2R_3 - R_1$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

$$R_3 \longrightarrow R_3 + R_2$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Rank of the coefficient matrix $r = 2$ = The No. of non zero rows.

So that the system have $n - r = 3 - 2 = 1$ L.I solution.

There is only one L.I eigen vector corrosponding to the eigen value $\lambda = 1$.

To determine this, we have to assign an arbitrary value for $n - r = 3 - 2 = 1$ variables.

The linear equations are

$$2x_1 + x_2 + x_3 = 0$$

$$3x_2 - 3x_3 = 0 \Longrightarrow x_2 - x_3 = 0.$$

Choose $x_3 = k_1$

$$x_2 = x_3 = k_1$$

$$2x_1 = -x_2 - x_3 = -2k_1$$

$$x_1 = -k_1$$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ where } k_1 \neq 0$$

$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is the linearly independent eigen vector corrosponding to the eigen value $\lambda = 1$. So that the geometric multiplicity of $\lambda = 1$ is 1.

Case (ii):- Eigen vector corrosponding to the eigen value $\lambda = 4$:-

For $\lambda = 4$, The system ① can be written as

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we reduce the coefficient matrix to echelon form by applying elementary row operations only and determine the rank of the coefficient matrix.

$$R_2 \longrightarrow R_2 + R_1, \quad R_3 \longrightarrow R_3 + R_1$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Rank of the coefficient matrix $r = 1$ = No. of non zero rows.

So that the system have $n - r = 3 - 1 = 2$ L.I solutions.

There are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 4$.

To determine this, we have to assign an arbitrary value for $n - r = 3 - 1 = 2$ variables

The linear equation is

$$-x_1 + x_2 + x_3 = 0$$

Choose $x_2 = k_2$

$x_3 = k_3$

$x_1 = x_2 + x_3 = k_2 + k_3$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_2 + k_3 \\ k_2 \\ k_3 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent eigen vectors corresponding to the eigen value $\lambda = 4$.

So that the ~~algebraic~~ geometric multiplicity of an eigen value $\lambda = 4$ is 2.

Since the geometric multiplicity of each eigen value of A coincides with the algebraic multiplicity

$\therefore$ A is a diagonalizable matrix.

Now we observe that the eigen vectors x_2 and x_3 are not pair-wise orthogonal.

Now we have to find the another linearly independent eigen vector x_2 of A corresponding to the eigen value $\lambda = 4$ such that x_1, x_2 and x_2, x_3 are pairwise orthogonal.

Let $x_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the another L.I eigen vector corresponding to the eigen value $\lambda = 4$.

x_1, x_2 are pairwise orthogonal if $-a + b + c = 0$ —— (2)

x_2, x_3 are pairwise orthogonal if $a + 0 \cdot b + c = 0$ —— (3)

Solving (2) and (3), we get

$$\begin{matrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{-1}$$

$x_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is the linearly independent eigen vector corresponding to the eigen value $\lambda = 4$ and is orthogonal to x_1 and x_3.

Now the eigen vectors $x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are pair-wise orthogonal.

$$\text{Modal matrix} = [x_1\ x_2\ x_3] = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$\|x_1\| = \sqrt{1+1+1} = \sqrt{3}$ $\quad \|x_2\| = \sqrt{1+4+1} = \sqrt{6}$

$\|x_3\| = \sqrt{1+0+1} = \sqrt{2}$

Normalized modal matrix $P = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \end{bmatrix}$

$$P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Here P is an orthogonal matrix.

By def. $PP^T = P^TP = I$

$$\Longrightarrow \quad P^{-1} = P^T$$

This matrix P will reduce the matrix A to the diagonal form which is given by $P^{-1}AP = D$ i.e $P^TAP = D$.

$$P^TAP = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is the spectral matrix.

Thus the orthogonal transformation $X = PY$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -1/\sqrt{6} & \frac{1}{\sqrt{2}} \end{bmatrix}$ transforms the given quadratic form to canonical form is given by.

$$Q = X^TAX$$

$$Q = (PY)^TA(PY)$$

$$= Y^T (P^T A P) Y$$

$$Q = Y^T D Y$$

$$Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Q = y_1^2 + 4y_2^2 + 4y_3^2$$

Rank of the Quadratic form $r = 3$

Index of the Quadratic form $s = 3$.

Signature of the Quadratic form $2s - r = 6 - 3 = 3$.

Nature of the Quadratic form is positive definite.

$\therefore$ The required orthogonal transformation which brings about the normal form is given by $X = PY$.

i.e
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \frac{-1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{6}} y_2 + \frac{1}{\sqrt{2}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_2$$

$$x_3 = \frac{1}{\sqrt{3}} y_1 - \frac{1}{\sqrt{6}} y_2 + \frac{1}{\sqrt{2}} y_3.$$

Maximize and Minimize the quadratic form $Q = X^T A X$ subject to $x^2+y^2+z^2 = 1$:-

Let $Q = X^T A X$ be the quadratic form.

step (i) :- Write the matrix of the given quadratic form.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ which is the symmetric matrix.

step (ii) :- The characteristic equation of A is $|A - \lambda I| = 0$

i.e $$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0.$$

Solve the characteristic equation, we get the eigen values of A.

step (iii) :- The eigen values of the matrix A are $\lambda_1, \lambda_2, \lambda_3$.

case (i) :- Let $\lambda = \max\{\lambda_1, \lambda_2, \lambda_3\}$

Suppose $\lambda = \lambda_1$

Find the eigen vector corresponding to the eigen value $\lambda = \lambda_1$

Let $X_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$

Find the normalized eigen vector $e_1 = \dfrac{X_1}{\|X_1\|}$

$$\|X_1\| = \sqrt{a_1^2 + b_1^2 + c_1^2}$$

$$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} \dfrac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \\ \dfrac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \\ \dfrac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \end{bmatrix}$$

Substitute the normalized eigen vector in given quadratic form,

we get maximum value of Q.

$\therefore$ Maximum value of Q = Maximum eigen value $= \lambda_1$.

Case(ii):- For minimize the quadratic form. Q.

Let $\lambda = \min\{\lambda_1, \lambda_2, \lambda_3\}$

Suppose $\lambda = \lambda_2$

Find the eigen vector x_2 corresponding to the eigen value. $\lambda = \lambda_2$

$$\text{Let } \quad X_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

Find the normalized eigen vector $e_2 = \dfrac{X_2}{\|X_2\|}$

$$\|X_2\| = \sqrt{a_2^2 + b_2^2 + c_2^2}$$

$$e_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} \dfrac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \\ \dfrac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \\ \dfrac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \end{bmatrix}$$

Substitute the normalized eigen vector in given quadratic form.

we get manimum value of Q.

$\therefore$ Minimum value of Q = Minimum eigen value $= \lambda_2$.

Nature of a Quadratic form $Q = X^T A X$ with the help of principal minors of the matrix A :—

The nature of a quadratic form can be determined from a study of the principal minors of the matrix of the quadratic form. In this method, the quadratic form need not be put in the canonical form.

Principal minors :—

Let $A = [a_{ij}]$ be a square matrix of order n. Then

$$M_1 = |a_{11}| \quad M_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad M_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdots M_n = |A|.$$

Working Rules :—

Case (i) :- A real quadratic form Q is positive definite if and only if all the principal minors of A are positive i.e $M_i > 0$ for all $i \leq n$.

Eg:- $Q = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_3 x_1$

The matrix A of the given quadratic form is given by

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$M_1 = |1| = 1 > 0 \quad M_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = 1 - \frac{1}{4} = \frac{3}{4} > 0.$$

$$M_3 = |A| = 1\left(1 - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{3}{4} - \frac{1}{8} - \frac{1}{8} = \frac{1}{2} > 0.$$

$$M_3 = > 0.$$

$$\therefore M_i > 0 \quad \forall i \leq 3.$$

$\therefore$ The nature of the given quadratic form is positive definite.

Case (ii) :- A real quadratic form Q is negative definite if and only if $M_1, M_3, M_5 \ldots$ are all negative and $M_2, M_4, M_6 \ldots$ are all positive.

i.e $(-1)^i M_i > 0$ for all i.

Eg :- $Q = -4x^2 - 2y^2 - 13z^2 - 4xy - 8yz - 4xz$.

The matrix A of the given quadratic form is given by

$$A = \begin{bmatrix} -4 & -2 & -2 \\ -2 & -2 & -4 \\ -2 & -4 & -13 \end{bmatrix}$$

$M_1 = |-4| = -4 < 0$ $\qquad M_2 = \begin{vmatrix} -4 & -2 \\ -2 & -2 \end{vmatrix} = 8 - 4 = 4 > 0.$

$M_3 = |A| = \left[-4(26-16) + 2(26-8) - 2(8-4)\right] = -40 + 36 - 8 = -12 < 0.$

Here $M_1 < 0$, $M_2 < 0$ $\quad M_2 > 0$.

nature of

∴ The given quadratic form is negative definite.

Case (iii) :- If some of the principal minors in case (i) are zero while the others are positive then the quadratic form Q is positive semi definite i.e $M_i \geq 0 \ \forall i \leq n$ and at least one $M_i = 0$.

Eg.- $Q = 10x^2 + 2y^2 + 5z^2 + 6yz - 10zx - 4xy$.

The matrix A of the given quadratic form is given by.

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

$M_1 = |10| = 10 > 0.$ $\qquad M_2 = \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = 20 - 4 = 16 > 0.$

$M_3 = |A| = 10[10-9] + 2[-10+15] - 5[-6+10] = 10 + 10 - 20 = 0$

Here $M_1 > 0$, $M_2 > 0$ and $M_3 = 0$

∴ The given quadratic form is positive semi definite.

nature of

Case (iv) :- If some of the principal minors in case (ii) are zero then Q is negative semi definite.

i.e. $(-1)^i M_i \geq 0 \ \forall i \leq n$ and at least one $M_i = 0$.

Eg :- $Q = -3x_1^2 - 3x_2^2 - 7x_3^2 - 6x_1x_2 - 6x_2x_3 - 6x_3x_1$.

The matrix of the quadratic form is

$$A = \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -7 \end{bmatrix}$$

$M_1 = |-3| = -3 < 0 \qquad M_2 = \begin{vmatrix} -3 & -3 \\ -3 & -3 \end{vmatrix} = 0$

$M_3 = |A| = -3[21-9] + 3[21-9] - 3[9-9] = 0$.

Here $M_1 < 0 \quad M_2 = 0 \quad M_3 = 0$.

$\therefore$ The given quadratic form Q is negative semi definite.
↳ nature of

Case (v) :- In all other cases, Q is indefinite.

Eg :- $Q = x^2 + 4y^2 + 4z^2 + 4xy + 6xz + 16yz$

The matrix of the quadratic form is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 4 \end{bmatrix}$$

$M_1 = |1| > 0 \quad M_2 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$.

$M_3 = |A| = 1(16 - \cancel{64}) - 2(8-24) + 3(16-12) = -48 + 32 + 12 = -8 < 0$

Here $M_1 > 0 \quad M_2 = 0 \quad M_3 < 0$.

$\therefore$ The nature of the given quadratic form is indefinite.

[illegible]

Here a_1 [illegible] a_2 [illegible] a_3 [illegible]

(ii) The given quadratic form is [illegible] semi-definite [illegible]

[illegible] the all other [illegible] is [illegible] definite

(iii) [illegible]

The matrix of the quadratic form is

[illegible]

[illegible]

The nature of the given quadratic form is indefinite.

Reduce the following quadratic forms to canonical form by an orthogonal transformation. Indicate its nature, rank, index and signature of the quadratic form. Also write the corresponding linear transformation which brings about the normal form.

(i) $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

Ans:- Rank = 3, Index = 3. Nature : Positive definite.

Eigen values : 1, 2, 4.

(ii) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$.

Ans:- Rank = 3, Index = 3. Nature Positive definite.

Eigen values : 2, 3, 6

(iii) $3x^2 - 2y^2 - z^2 + 12yz + 8zx - 4xy$.

Ans:- Rank = 3 Index = 2 Nature: Indefinite.

Eigen values : 3, 6, −9

(iv) $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4zx$.

Ans:- Rank = 2 Index = 2 Nature : Positive semi definite.

Eigen values: 0, 3, 15

(v) $3x^2 + 2y^2 + 3z^2 - 2xy + 2yz$.

Ans:- Rank = 3. Index = 3 Nature : Positive definite

Eigen values. 3, 1, 4.

(vi) $7x^2 + 5y^2 + 6z^2 - 4xz - 4yz$

Ans. Rank = 3, Index = 3 Nature : Positive definite.

Eigen values : 3, 6, 9.

(vii) $3x^2 + 2y^2 - 4xz$. Eigen values : −1, 2, 4

Ans:- Rank = 3, Index = 2 Nature Indefinite.

(viii) $6x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ Eigen values : 6, 2, 4

Ans:- Rank = 3, Index = 3 Nature Positive definite.

Reduce the following quadratic forms to canonical form by an ortho-gonal transformation. Indicate Rank, index, nature and signature of the quadratic form. Also indicate the matrix of the transformation.

(i) $2x^2+2y^2+2z^2-2xy-2yz+2zx$.

Ans:- Rank $=3$ Index $=3$ Nature: Positive definite.

Eigen values: $1, 1, 4$

(ii) $2xy+2yz+2zx$

Ans: Rank $=3$ Index $=1$ Nature: Indefinite.

Eigen values: $-1, -1, 4$.

(iii) $3x^2+3y^2+3z^2+2xy+2xz-2yz$

Ans:- Rank $=3$, Index $=3$ Nature: Positive definite.

Eigen values: $1, 4, 4$

(iv) $2x_1x_2+2x_1x_3-2x_2x_3$.

Ans:- Rank $=3$, Index $=2$ Nature: Indefinite.

Eigen values $1, 1, -2$

(v) $6x^2+3y^2+3z^2-4xy-2yz+4xz$, Eigen values: $2, 2, 8$.

Ans:- Rank $=3$ Index $=3$ Nature Positive definite.

(vi) $2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_1x_3-2x_2x_3$

Ans: Rank $=2$ Index $=2$ Nature: Positive semi definite.

Eigen values: $0, 3, 3$

(vii) $-3x_1^2-3x_2^2-3x_3^2-2x_1x_2-2x_1x_3+2x_2x_3$

Ans:- Rank $=3$, Index $=0$ Nature: Nagative definite.

Eigen values: $-4, -4, -1$.

(viii) $x^2+y^2+z^2+4yz+4xy+4zx$.

Ans.- Rank $=3$, Index $=2$ Nature: Indefinite.

(1) Find the maximum and minimum values of $f(x,y) = 3x^2 - 3y^2 + 8xy$ subject to $x^2 + y^2 = 1$.

Ans:- Max. of $f = 5$, Min. of $f = -5$.

(2) Find the maximum and minimum values of $f(x,y,z) = 3x^2 + 3z^2 + 2y^2 + 2xz$ subject to $x^2 + y^2 + z^2 = 1$.

Ans:- Max. of $f = 4$. Min. of $f = 2$

(3) Find the maximum and minimum values of $f(x,y,z) = 10x^2 + 2y^2 + 5z^2 - 4xy - 10xz + 6yz$ subject to $x^2 + y^2 + z^2 = 1$.

Max. of $f = 14$. Min. of $f = 0$.

(4) Find the maximum and minimum values of $2x^2 + 5y^2 + 3z^2 + 4xy$. subject to $x^2 + y^2 + z^2 = 1$.

Max. of $f = 6$ Min. of $f = 1$.

(1) Identify the nature of the following quadratic forms. Also write Rank, Index and signature of the quadratic form.

(a) $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$.

Ans:- Nature: +ve semi definite, Index = 1, Rank = 1.

(b) $x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$.

Ans:- Nature: Indefinite Index = 1 Rank = 2.

(c) $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$.

Ans:- Nature: Positi definite Index = 3 Rank = 3.

(d) $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz$.

Ans:- Nature: +ve semi definite Index = 2 Rank = 2.

Reduce the following quadratic forms to canonical form by Lagrange's method. Also write the corresponding linear transformation. Find its rank, index nature and signature of the quadratic form.

1(a) $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

Ans: Rank = 3, Nature - Indefinite Index = 2.

1(b) $2x_1^2 + 7x_2^2 + 5x_3^2 - 8x_1x_2 - 10x_2x_3 + 4x_1x_3$.

Ans: Rank = 3 Nature: Indefinite Index = 2.

1(c) $x_1^2 + 3x_2^2 + x_3^2 + 2x_1x_2 + 4x_2x_3 + 6x_1x_3$.

Ans:- Rank = 3 Nature: Indefinite Index = 2.

1(d) $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_3x_1 - 4x_2x_3$.

Ans:- Rank = 1. Nature: Positive semi definite Index = 1.

(e) $x_1^2 + 6x_2^2 + 18x_3^2 + 4x_1x_2 + 8x_1x_3 - 4x_2x_3$.

Ans: Rank = 3 Nature: Indefinite Index = 2.

(f) $x^2 + y^2 + z^2 - 2xy + 4xz + 4yz$

Ans:- Rank = 3 Nature: Indefinite Index = 2.

1(g) $x_1^2 - 4x_2^2 + 5x_3^2 + 2x_1x_2 - 4x_1x_3 + 2x_4^2 - 6x_3x_4$

Ans: Rank = 4 Nature: Indefinite Index = 2.

1(h) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

Ans:- Rank = 3 Nature: Positive definite Index = 3.

MODULE -III

ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

1. Solution of Algebraic and Transcendental Equations

Algebraic Equation :—

An equation is of the form $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$ is called an algebraic equation of n^{th} degree where $a_0, a_1, a_2, \cdots a_n$ are real numbers, n is a non -ve integer and $a_0 \neq 0$.

Eg:- $5x^2 - 7x + 8 = 0$, $4x^3 - 8x + 9 = 0$ are algebraic equation of degree 2, 3.

Transcendental Equation :—

If $f(x)$ contains the some other functions namely trigonometric, logarithmic, exponential etc then the eqn $f(x) = 0$ is called transcendental equation.

Eg $xe^x = \sin x$, $3x - \log_{10} x = 8$.

Zero (or) Root of a Equation :—

A number α (real or complex) is called a root (or solution) of the equation $f(x) = 0$ if $f(\alpha) = 0$. We can also say that α is a zero of the function $f(x)$.

Geometrically the root of $f(x) = 0$ is the value of x at which the graph of $f(x)$ meet the x-axis

A Polynomial Equation of degree n will have exactly n roots real, or complex, simple or multiple.

A number α is a simple root of $f(x) = 0$ if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$

Then we can write $f(x) = (x - \alpha) g(x)$, $g(\alpha) \neq 0$.

A number α is a multiple root of multiplicity m of $f(x)=0$ if $f(\alpha)=f'(\alpha)=f''(\alpha)=\dots=f^{(m-1)}(\alpha)=0$ and $f^{(m)}(\alpha)\neq 0$ Then $f(x)$ can be written as $f(x)=(x-\alpha)^m g(x)$, $g(\alpha)\neq 0$.

(2)

Exact and Approximate Numbers :—

Exact Numbers are those which have definite value.

Eg:- $2, 7, 9, \frac{1}{2}, \frac{1}{15}, \pi, e$ etc.

Approximate numbers are those that represent that exact numbers to a certain degree of accuracy

Eg:- $\sqrt{2}\simeq 1.414$ $e\simeq 2.7183$, $\pi\simeq 3.142\dots$ are not exact. since they contain infinitely many digits. The numbers obtained by retaining a few digits are called Approximate numbers.

Common Method of Rounding :—

This method is commonly used in accounting. This method is also known as symmetric arithmetic rounding or round half up (symmetric implementation).

→ Decide which is the last digit to keep.

→ Increase it by 1 if the next digit 5 or more (rounding up).

→ Leave the same if the next digit is 4 or less (rounding down).

Eg:- (i) 3.044 rounded to hundredths is 3.04 because the next digit is 4, which is less than 5.

(ii) 3.046 rounded to hundredths is 3.05 because the next digit is 6 which is more than 5.

(iii) 3.0447 rounded to hundredths is 3.04 because the next digit 4, is less than 5.

Significant figures :—

All the digits 1, 2, 3 ... 9 are significant figures, '0' may or may not be significant figure. It depends on the context in which zero has been used.

Zeroes concept :-

(i) Zeroes appearing between non zero digits are significant

Eg:- 60.8 has three significant figures.

3900 8 has five significant figures.

(ii) Zeroes appearing in front of non zero digits are not significant

Eg:- 0.093827 have 5 significant figures

0.0008 has one significant figure.

(iii) Zeroes at the end of a number and to the right of a decimal are significant.

Eg:- 35.00 has four significant figures.

8,000.000000 has ten significant figures.

(iv) Zeroes at the end of a number without a decimal point may or may not be significant and are therefore ambiguous.

Eg:- 1,000 could have between one and four significant figures

This ambiguity could be resolved by placing a decimal after the number, e.g writing "1,000." to indicate specifically that four significant figures are meant, but this is a non-standard usage. To specify unambiguously how many significant figures are implied, scientific notation can be employed.

→ 1×10^3 has one significant figure, while

→ 1.000×10^3 has four.

Rules for counting Significant Digits :—

(1) Always count non zero digits.

Eg :- 21 has two significant figures while 8.926 has four.

(2) Never count leading zeroes.

Eg:- 021 and 0.021 both have two significant figures.

(3) Always count zeroes which fall somewhere between two non zero digits.

Eg.- 20.8 has 3 significant figures while 0.00104~~0~~009 has seven.

(4) Count trailing zeros iff the number contains a decimal point.

Eg:- 210 and 210000 both have 2 significant figures while 210. has three 210.00 has five.

(5) For numbers expressed in scientific notation, ignore the exponent apply Rules 1-4 to the mantissa.

Eg:- 4.2010×10^{28} has five significant figures.

Rounding off :—

cutting off of some of the end digits of a number and retaining a fixed number of digits in numerical computation is called rounding off.

Rules of Rounding off numbers :—

(1) In rounding off numbers, the last figure kept should be unchanged if the first figure dropped is less than 5.

Eg:- If only one decimal is to be kept then 6.422 becomes 6.4.

(2) In rounding off numbers the last figure kept should be increased by 1 if the first figure dropped is greater than 5.

Bisection Method :—

This method is based on the repeated application of the intermediate value theorem to obtain an approximation to the root. Suppose that a root of $f(x)=0$ lies in the interval $I_0=(a_0, b_0)$, that is $f(a_0)f(b_0)<0$.
We bisect this interval and obtain $c_1=\frac{a_0+b_0}{2}$ Then, the root lies in the interval $I_1=(a_0, c_1)$ if $f(a_0)f(c_1)<0$.
otherwise, it lies in the interval $I_1=(c_1, b_0)$.
Thus the length of the interval I_1, is one half of that of I_0.
Continuing this procedure, we obtain a nested set of subintervals $I_0 \supset I_1 \supset I_2 \cdots$ such that each of these subintervals contains the root.
After repeating the bisection procedure n times, we obtain an interval I_n of length $(b_0-a_0)/2^n$. which contains the root.
The mid-point of the last interval is taken as the required approximation to the root.
Note that the method does not use the value of $f(x)$, but only its sign. Hence if an accuracy for the root is prescribed, we can determine in advance for all equations, the number of iterations.
After the nth iteration, we have $\frac{b_0-a_0}{2^n} \le \varepsilon$.

Taking the logarithms, we get

$$\log\left(\frac{b_0-a_0}{2^n}\right) \le \log \varepsilon$$

$$\log(b_0-a_0) - \log 2^n \le \log\varepsilon$$

$$\log(b_0-a_0) - n\log 2 \le \log\varepsilon$$

$$n\log 2 \ge \log(b_0-a_0) - \log\varepsilon$$

$$\therefore n \ge \text{int}\left[\frac{\log(b_0-a_0)-\log\varepsilon}{\log 2}\right]$$

Where Int [...] stands for the next integer.

For example if $b_0 - a_0 = 1$ and $\varepsilon = 10^{-2}$ we get $n \geq 7$. (6)

that is seven iterations are required.

Note:- (i) The cost of the method is one function evaluation per iteration

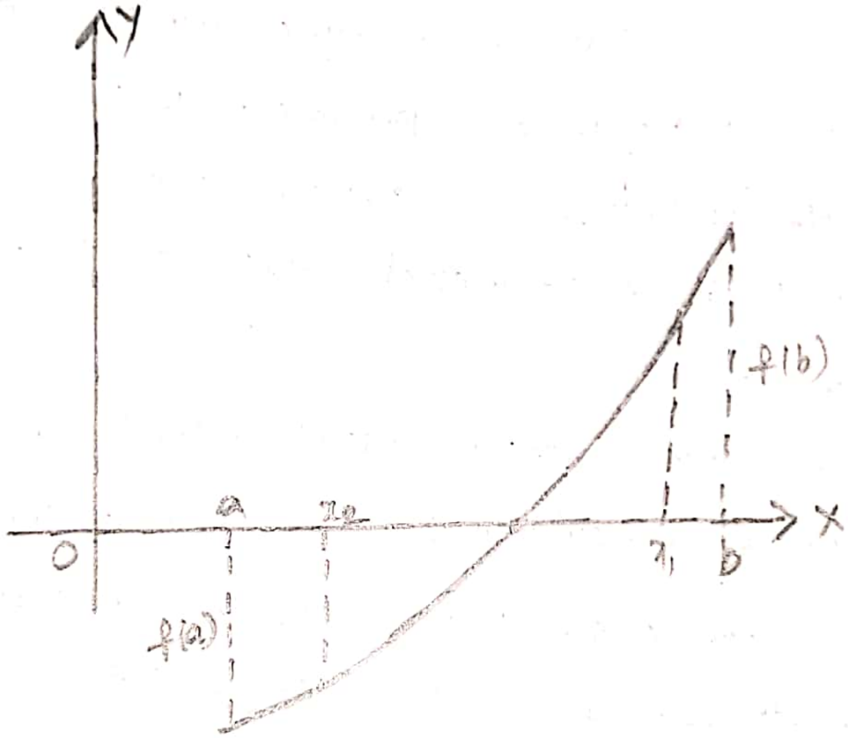
(ii) The method never fails as the root lies in the interval being considered

(iii) The value of $f(x_k)$ is not used, but only its sign.

(iv) At each iteration, the length of the interval is reduced by a factor of 2.

(v) The method has linear convergence.

(vi) The method is very slow, if high accuracy is required.



Eg:- If only two decimals are to be kept then 6.4872 becomes 6.49 similarly 6.997 becomes 7.00 .

3

(3) In rounding off numbers if the first figure dropped is 5 and all the figures following the five are zero or if there are no figures after the 5, then the last figure kept should be unchanged if that last figure is even.

Eg: If only one decimal is to be kept then 6.65 becomes 6.6.

Eg: If only two decimals are to be kept then 7.485 becomes 7.48 .

Rule(4) :- In rounding off numbers, if the first figure dropped is 5. and all the following the five are zero or if there are no figures after the 5, then the last figure kept should be increased by 1 if that last figure is odd.

Eg :- If only two decimals are to be kept then 6.755000 becomes 6.76 .

Eg :- if only two decimals are to be kept 8.995 becomes 9.00.

(5) In rounding off numbers, if the first figure dropped is 5 and there are any figures following the five that are zero, then the last figure kept should be increased by 1 .

Eg: If only one decimal is to be kept then 6.6501 becomes 6.7.

If only two decimals are to be kept then 7.4852007 becomes 7.49 .

Number	Number of decimal places desired	Last figure to be kept	First figure to be dropped	Last figure kept and last number becomes
6.422	1	6.4	6.42	6.4
6.4872	2	6.48	6.487	6.49
6.997	2	6.99	6.997	7.00.
6.6500	1	6.6	6.65	6.6
7.485	2	7.48	7.485	7.48
6.755000	2	6.75	6.755	6.76
8.995	2	8.99	8.995	9.00.
6.6501	1	6.6	6.65	6.7
7.4852007	2	7.48	7.485	7.49.

Error :- If the difference b/n the Exact number and Approximate number are called the Error that means Exact is denoted by X, approximate is denoted by x. $E = X - x$.

Absolute Error :- Absolute Error is denoted by E_A and $E_A = |X - x|$ that means $E_A = |E|$.

Maximum Error is called Absolute Error.

Relative Error :- Relative Error is denoted by E_R and $E_R = \frac{E_A}{|X|}$

$$E_R = \frac{|X - x|}{|X|}$$

Percentage Error :- Percentage Error is denoted by E_P and

$E_P = E_R \times 100$. $\quad E_P = \frac{|X-x|}{|X|} \times 100$

The Methods of finding the root of $f(x)=0$ are classified as

(i) Direct Methods

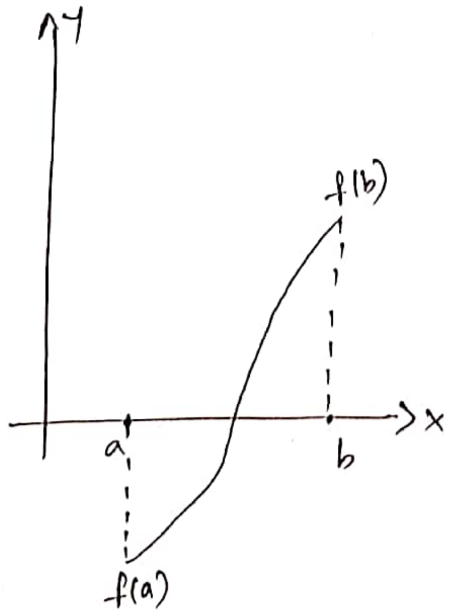
(ii) Numerical Methods.

Direct Methods give the exact values of all the roots in a finite number of steps.

Numerical Methods are based on the idea of successive approximations. In these methods we start with one or two initial approximations to the root and obtain a sequence of approximations $x_0, x_1, x_2, \dots x_n$ which in the limit as $n \to \infty$ converge to the exact root $x = a$.

There are no direct methods for solving higher degree algebraic Equations or transcendental Eqn's such eqns can be solved by Numerical methods.

<u>Theorem</u> :- Let the function $f(x)$ be continuous on $[a, b]$. Let $f(a) < 0$ and $f(b) > 0$ then $f(x) = 0$ will have atleast one root between a and b.



Bisection Method :—

Step 1 :- Consider the equation $f(x)=0$

Identity two points $x=a$ and $x=b$ such that $f(a)$ and $f(b)$ have opposite signs. Let $f(a)$ be $-ve$ and $f(b)$ be $+ve$.

Then there will be a root of the equation $f(x)=0$ in b/n a and b.

Step 2 :- Find $x_1 = \frac{a+b}{2}$, calculate $f(x_1)$

If $f(x_1)=0$ then x_1 becomes the root of the equation $f(x)=0$.

otherwise,

(a) If $f(x_1) < 0$ then the root lies between x_1 and b.

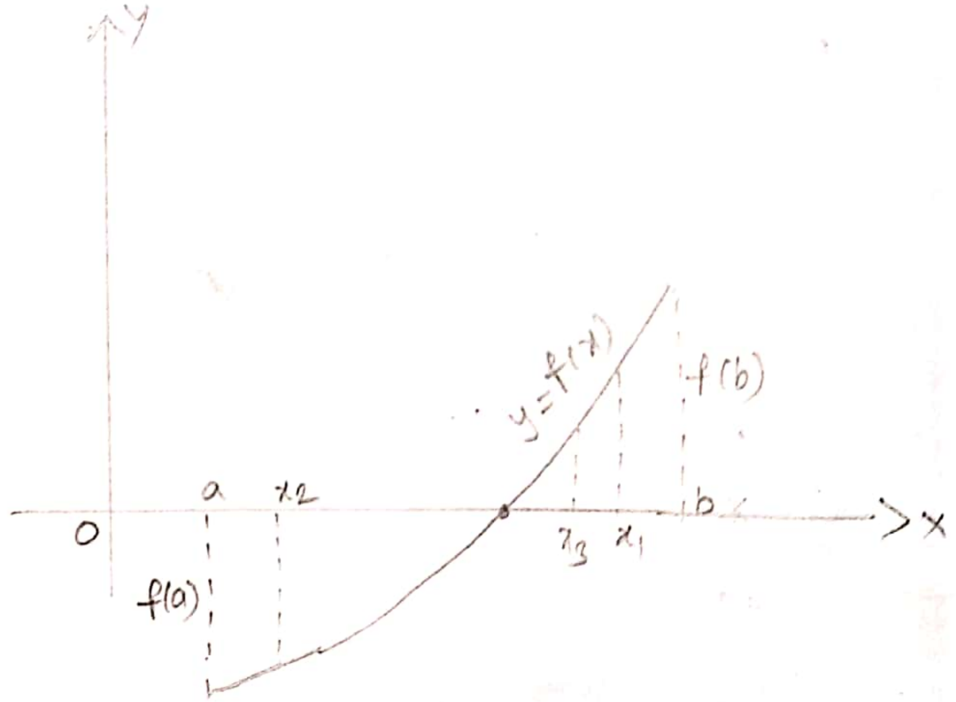
Find $x_2 = \frac{x_1+b}{2}$, calculate $f(x_2)$ and so on.

else

(b) If $f(x_1) > 0$ then the root lies between a and x_1

Find $x_3 = \frac{a+x_1}{2}$, calculate $f(x_3)$ and so on.

We proceed in this way until the two successive approximations are approximately equal.



(1) Find a real root of the equation $x \log_{10} x = 1.2$ which lies between 2 and 3 by bisection method upto 5 approximation.

Sol:- Let $f(x) = x \log_{10} x - 1.2$

$x = 2, \quad f(2) = 2\log_{10} 2 - 1.2 = 0.60206 - 1.2 = -0.59794 < 0.$

$x = 3, \quad f(3) = 3\log_{10} 3 - 1.2 = 1.43136 - 1.2 = 0.23136 > 0.$

since $f(2)$ and $f(3)$ are of opposite signs

$\therefore$ The given equation have root between 2 and 3.

First Approximation :-

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(x_1) = f(2.5) = 2.5\log_{10}(2.5) - 1.2 = 0.99485 - 1.2 = -0.2051 < 0.$$

We observe that $f(2.5)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.5 and 3.

Second Approximation :-

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$$f(x_2) = f(2.75) = (2.75)\log_{10} 2.75 - 1.2 = 1.20816 - 1.2 = 0.00816 > 0$$

We observe that $f(2.5)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.5 and 2.75.

Third Approximation :-

$$x_3 = \frac{2.5 + 2.75}{2} = 2.625$$

$$f(x_3) = f(2.625) = (2.625)\log_{10} 2.625 - 1.2 = 1.1002 - 1.2 = -0.09978 < 0$$

We observe that $f(2.625)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.625 and 2.75

Fourth Approximation :—

$$x_4 = \frac{2.625 + 2.75}{2} = 2.6875$$

$f(x_4) = f(2.6875) = 2.6875 \log_{10} 2.6875 - 1.2 = 1.153874 - 1.2 = -0.0461 < 0.$

We observe that $f(2.6875)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.6875 and 2.75

Fifth Approximation :—

$$x_5 = \frac{2.6875 + 2.75}{2} = 2.71875$$

$$f(x_5) = f(2.71875) = (2.71875)\log_{10}(2.71875) - 1.2 = 1.18094 - 1.2$$
$$= -0.01906.$$

$\therefore$ An approximate root of the given equation $x\log_{10} x - 1.2$ is

2.71875.

→ Find a positive root of the equation $x^3-4x-9=0$ using bisection method in five stages and correct to four decimal places. (4)

Sol: Let $f(x) = x^3-4x-9$.

$x=0$ $f(0) = 0-0-9 = -9 < 0$

$x=1$ $f(1) = 1-4-9 = -12 < 0$

$x=2$ $f(2) = 8-8-9 = -9 < 0$

$x=3$ $f(3) = 27-12-9 = 6 > 0$

We note that $f(2)$ and $f(3)$ are of opposite signs.

∴ The given equation $x^3-4x-9=0$ does have a real root between 2 and 3.

First Approximation :—

$$x_1 = \frac{2+3}{2} = 2.5$$

$f(x_1) = f(2.5) = (2.5)^3 - 4(2.5) - 9 = 15.625 - 10 - 9 = -3.375 < 0.$

We observe that $f(2.5)$ and $f(3)$ are of opposite signs.

∴ The root lies between 2.5 and 3.

Second Approximation :—

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$f(x_2) = f(2.75) = (2.75)^3 - 4(2.75) - 9 = 20.796875 - 11 - 9 = 0.796875 > 0.$

We observe that $f(2.5)$ and $f(2.75)$ are of opposite signs.

∴ The root lies between 2.5 and 2.75.

Third Approximation :—

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

$f(x_3) = f(2.625) = (2.625)^3 - 4(2.625) - 9$

$= 18.08789063 - 10.5 - 9 = -1.41211 < 0.$

We observe that $f(2.625)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.625 and 2.75

Fourth Approximation :-

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

$$f(x_4) = f(2.6875) = (2.6875)^3 - 4(2.6875) - 9$$

$$= 19.41088867 - 10.75 - 9 = -0.33911 < 0$$

We observe that $f(2.6875)$ and $f(2.75)$ are of opposite signs.

$\therefore$ The root lies between 2.6875 and 2.75.

Fifth Approximation :-

$$x_5 = \frac{2.6875+2.75}{2} = 2.71875$$

This is an approximate value of the required root of the given equation, obtained at the fifth stage of bisection.

$\therefore$ An approximate root of given equation is $x^3-4x-9=0$ is 2.71875.

→ Using the bisection method find an approximate root of the equation $x - \cos x = 0$ that lies between 0.5 and 1 (Here x is in radians) carry out five steps of approximations.

Sol: Let $f(x) = x - \cos x$

$x = 0.5 \quad f(0.5) = 0.5 - \cos(0.5) = 0.5 - 0.87758 = -0.37758 < 0.$

$x = 1 \quad f(1) = 1 - \cos(1) = 1 - 0.540302 = 0.4597 > 0.$

We note that $f(0.5)$ and $f(1)$ are of opposite signs

∴ The given equation does have a real root between 0.5 and 1.

First Approximations :—

$$x_1 = \frac{0.5+1}{2} = 0.75$$

$f(x_1) = f(0.75) = 0.75 - \cos(0.75) = 0.75 - 0.73169 = 0.01831 > 0.$

We observe that $f(0.5)$ and $f(0.75)$ are of opposite signs.

∴ The root lies between 0.5 and 0.75.

Second Approximations :—

$$x_2 = \frac{0.5+0.75}{2} = 0.625$$

$f(x_2) = f(0.625) = 0.625 - \cos(0.625) = 0.625 - 0.81096 = -0.18596 < 0.$

We observe that $f(0.625)$ and $f(0.75)$ are of opposite signs.

∴ The root lies between 0.625 and 0.75

Third Approximations :-

$$x_3 = \frac{0.625+0.75}{2} = 0.6875$$

$f(x_3) = f(0.6875) = 0.6875 - \cos(0.6875) = 0.6875 - 0.7728 = -0.0853 < 0.$

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We observe that $f(0.6875)$ and $f(0.75)$ are of opposite signs.

$\therefore$ The root lies between 0.6875 and 0.75.

<u>Fourth</u> <u>Approximation</u> :—

$$x_4 = \frac{0.6875+0.75}{2} = 0.71875$$

$f(x_4) = f(0.71875) = 0.71875 - \cos(0.71875) = 0.71875 - 0.75263 = -0.033 < 0.$

We observe that $f(0.71875)$ and $f(0.75)$ are of opposite signs.

$\therefore$ The root lies between 0.71875 and 0.75.

<u>Fifth</u> <u>Approximation</u> :-

$$x_5 = \frac{0.71875+0.75}{2} = 0.734375.$$

This is an approximate value of the required root of the given equation obtained at the fifth stage of bisection.

$\therefore$ An approximate root of given equation $x - \cos x = 0$ is 0.734375.

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$f(x_7) = f(1.11328125) = 1.11328125 \sin(1.11328125) - 1 = 0.99878 - 1$
$= -0.00216 < 0$

We observe that 1.11328125 and 1.1171875 are of opposite signs

<u>Eighth</u> <u>Approximation</u> :-

$\begin{array}{ccc} -ve & +ve & +ve \\ f(1.11328125)<0 & f(1.115234375)>0 & f(1.1171875)>0 \\ 1.11328125 & 1.115234375 & 1.1171875 \end{array}$

$$x_8 = \frac{1.11328125 + 1.1171875}{2} = 1.115234375$$

$$f(x_8) = f(1.115234375) = (1.115234375) \sin(1.115234375) - 1$$
$$= 1.00149 - 1 = 0.00149 > 0.$$

We observe that $f(1.11328125)$ and $f(1.115234375)$ are of opposite signs.

$\therefore$ The root lies between 1.11328125 and 1.115234375.

<u>Ningth</u> <u>Approximation</u> :-

$\begin{array}{ccc} -ve & +ve & +ve \\ f(1.11328125)<0 & f(1.114257813)>0 & f(1.115234375)>0 \\ 1.11328125 & 1.114257813 & 1.115234375 \end{array}$

$$x_9 = \frac{1.11328125 + 1.115234375}{2} = 1.114257813.$$

$f(x_9) = f(1.114257813) = 1.114257813 \sin(1.114257813) - 1 = 1.00014 - 1 = 0.0001 > 0.$

We observe that $f(1.11328125)$ and $f(1.114257813)$ are of opposite signs.

$\therefore$ The root lies between 1.11328125 and 1.114257813.

<u>Tenth</u> <u>Approximation</u> :-

$\begin{array}{ccc} -ve & -ve & +ve \\ f(1.11328125)<0 & f(1.113769532)<0 & f(1.114257813)>0 \\ 1.11328125 & 1.113769532 & 1.114257813 \end{array}$

$$x_{10} = \frac{1.11328125 + 1.114257813}{2} = 1.113769532$$

$f(x_{10}) = f(1.113769532) = 1.113769532 \sin(1.113769532) - 1 = 0.99646 - 1 = -0.0005 < 0.$

We observe that $f(1.113769532)$ and $f(1.114257813) > 0$ are of opposite signs.

$\therefore$ The root lies between 1.113769532 and 1.114257813.

Eleventh Approximation :-

$$x_{11} = \frac{1.1137695312 + 1.1142578125}{2} = 1.1140136719$$

$f(x_{11}) = f(1.1140136719) = 1.1140136719 \sin(1.1140136719) - 1 = 0.9998 - 1 = -0.00019 < 0$

We observe that 10^{th} and 11^{th} approximations are approximately equal.

This is an approximate value of the required root of the given equation obtained at the 11^{th} stage of bisection.

$\therefore$ An approximate root of given equation $x\sin x - 1 = 0$ is 1.1140.

BISECTION METHOD

1. Explain Bisection method with geometrical interpretation.
2. Find an approximate value of the root of the equation $x^3 - x - 11 = 0$ that lies between 2 and 3 using bisection method.
3. Find by using the bisection method, an approximate root of the equation $x^4 - x^3 - 2x^2 - 6x - 4 = 0$, that lies in the interval $(2, 2.75)$.
4. Find the nagative root of the equation $x^3 - 4x + 9 = 0$ using bisection method.
5. Find the nagative root of the equation $x^3 - 3x + 4 = 0$ using bisection method.
6. Find an approximate root of the equation. $x + \log_e^x = 5$ in $(3.2, 4)$ using bisection method.
7. Find the root of the equation $x = e^{-x}$ in $(0,1)$ using bisection method.
8. Find the root of the equation $2x - \log_{10}^x = 7$ in $(3.5, 4)$ using bisection method.
9. Find the root of the equation $x^2 - \log x = 12$ in $(3,4)$ by bisection method.
10. Find the root of the equation $e^x \sin x = 1$ in $(2,3)$ in radians using bisection method.
11. Using bisection method, find an approximate root of the equation. $\log x - \cos x = 0$ in $(1,2)$ in radians.
12. Using bisection method, find the root of $\sin x - 2x + 1 = 0$.
13. Find the root of the equation $x^2 + x - \cos x = 0$ using bisection method.

14. Find an approximate value of square root of 15 using bisection method.

15 Find an approximate value of cube root of 9 using bisection method.

16. Find an approximate value of 5th root of 5 using bisection method.

17 Prove that the order of convergence of Bisection method is linear

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→ By using the bisection method, find an approximate root of the equation $\sin x = \frac{1}{x}$ that lies between $x=1$ and $x=1.5$ (measured in radians) and correct to four decimal places.

Sol:- Given that $\sin x = \frac{1}{x}$ i.e $x\sin x - 1 = 0$.

Let $f(x) = x\sin x - 1$.

$x=1$ $\quad f(1) = \sin 1 - 1 = 0.84147 - 1 = -0.1585 < 0$.

$x=1.5$ $\quad f(1.5) = 1.5\sin(1.5) - 1 = 1.4962 - 1 = 0.4962 > 0$.

We note that $f(1)$ and $f(1.5)$ are of opposite signs

∴ The given equation $x\sin x - 1 = 0$ has a root between 1 and 1.5.

First Approximation :-

$-ve$ $f(1)<0$ $+ve$ $f(1.25)>0$ $+ve$ $f(1.5)>0$

1   1.25   1.5

$$x_1 = \frac{1+1.5}{2} = 1.25$$

$f(x_1) = f(1.25) = 1.25\sin(1.25) - 1 = 1.18623 - 1 = 0.18623 > 0$.

We observe that $f(1)$ and $f(1.25)$ are of opposite signs.

∴ The root lies between 1 and 1.25.

Second Approximation :-

$-ve$ $f(1)<0$ $+ve$ $f(1.125)>0$ $+ve$ $f(1.25)>0$

1   1.125   1.25

$$x_2 = \frac{1+1.25}{2} = 1.125.$$

$f(x_2) = f(1.125) = 1.125\sin(1.125) - 1 = 1.01505 - 1 = 0.01505 > 0$.

We observe that $f(1)$ and $f(1.125)$ are of opposite signs.

∴ The root lies between 1 and 1.125.

Third Approximation :-

$-ve$ $f(1)<0$ $-ve$ $f(1.0625)<0$ $+ve$ $f(1.125)>0$

1   1.0625   1.125

$$x_3 = \frac{1+1.125}{2} = \frac{2.125}{2} = 1.0625.$$

$f(x_3) = f(1.0625) = 1.0625\sin(1.0625) - 1 = 0.92817 - 1 = -0.077183 < 0$.

We observe that $f(1.0625)$ and $f(1.125)$ are of opposite signs.

$\therefore$ The root lies between 1.0625 and 1.125.

Fourth Approximation :—

$-ve$ $f(1.0625)<0$ $-ve$ $f(1.09375)<0$ $+ve$ $f(1.125)>0$

1.0625   1.09375   1.125

$$x_4 = \frac{1.0625+1.125}{2} = \frac{2.1875}{2} = 1.09375$$

$$f(x_4) = f(1.09375) = 1.09375\sin(1.09375)-1 = 0.9716-1 = -0.028 < 0$$

We observe that $f(1.09375)$ and $f(1.125)$ are of opposite signs.

$\therefore$ The root lies between 1.09375 and 1.125.

Fifth Approximation :-

$-ve$ $f(1.09375)<0$ $-ve$ $f(1.109375)<0$ $+ve$ $f(1.125)>0$

1.09375   1.109375   1.125

$$x_5 = \frac{1.09375+1.125}{2} = \frac{2.21875}{2} = 1.109375$$

$$f(x_5) = f(1.109375) = 1.109375\sin(1.109375)-1 = 0.9934-1 = -0.0066 < 0$$

We observe that $f(1.109375)$ and $f(1.125)$ are of opposite signs.

$\therefore$ The root lies between 1.109375 and 1.125.

Sixth Approximation :-

$-ve$ $f(1.109375)<0$ $+ve$ $f(1.1171875)>0$ $+ve$ $f(1.125)>0$

1.109375   1.1171875   1.125

$$x_6 = \frac{1.109375+1.125}{2} = \frac{2.234375}{2} = 1.1171875$$

$$f(x_6) = f(1.1171875) = 1.1171875\sin(1.1171875)-1 = 1.0042-1 = 0.0042 > 0$$

We observe that $f(1.1171875)$ and $f(1.109375)$ are of opposite signs.

$\therefore$ The root lies between 1.109375 and 1.1171875.

Seventh Approximation : -

$-ve$ $f(1.109375)<0$ $-ve$ $f(1.11328125)<0$ $f(1.1171875)>0$

1.109375   1.11328125   1.1171875

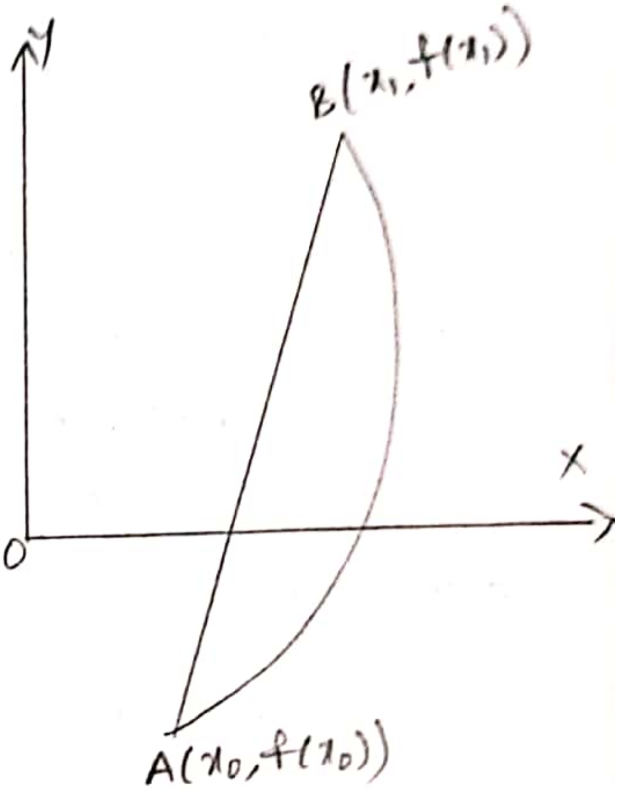
$$x_7 = \frac{1.109375+1.1171875}{2} = \frac{2.2265625}{2} = 1.11328125$$

Regula Falsi Method Formula :-

Consider the equation $f(x)=0$

Let x_0 and x_1 be two values of x such that $f(x_0)$ and $f(x_1)$ are opposite signs. since the graph of $y=f(x)$ crosses x-axis. the root must lies between x_0 and x_1

The chord joining $A(x_0,f(x_0))$ and $B(x_1,f(x_1))$ meets x-axis



Equation of the line AB is

$$y-f(x_0)=\frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)$$

The point of intersection of the line with x-axis will be the first approximation of the root of $f(x)=0$.

Let it be $(x_2,0)$. At this point $y=0$.

$$0-f(x_0)=\frac{f(x_1)-f(x_0)}{x_1-x_0}(x_2-x_0)$$

$$x_2-x_0=\frac{-f(x_0)(x_1-x_0)}{f(x_1)-f(x_0)}$$

$$x_2=x_0-\frac{f(x_0)(x_1-x_0)}{f(x_1)-f(x_0)}$$

$$=\frac{x_0f(x_1)-x_0f(x_0)-x_1f(x_0)+x_0f(x_0)}{f(x_1)-f(x_0)}$$

$$x_2=\frac{x_0f(x_1)-x_1f(x_0)}{f(x_1)-f(x_0)}$$

If $f(x_0)$ and $f(x_2)$ are of opposite signs then the root lies between x_0 and x_2. Put $x_1=x_2$, then we get next approximation.

If $f(x_1)$ and $f(x_0)$ are of opposite signs then the root lies between x_1 and x_2

Put $x_0 = x_2$ we get next approximation.

We proceed in this way till the two successive approximations are approximately equal.

The Regula Falsi

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}.$$

Working Procedure :-

Consider the equation $f(x) = 0$.

Step 1 :- Consider two initial approximations to the root are x_0, x_1.

Find $f(x_0)$, $f(x_1)$. Assume that $f(x_0) < 0$, $f(x_1) > 0$.

$\therefore$ The root lies between the points x_0, x_1.

Step 2 :- Find x_2 using the formula. $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$.

Calculate $f(x_2)$. If suppose $f(x_2) > 0$.

$\therefore$ The root lies between the points x_0, x_2.

Step 3 :- Replace x_0 by x_1.

Find x_3 using the formula $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$.

Calculate $f(x_3)$. If suppose $f(x_3) < 0$.

$\therefore$ The root lies between the points x_2, x_3.

Step 4 :- We proceed in this way until the two successive approximations are approximately equal.

Regula Falsi Method

Let us consider the equation $f(x)=0$

Consider two initial approximate values x_0 and x_1 near the required root so that $f(x_0)$ and $f(x_1)$ have different signs

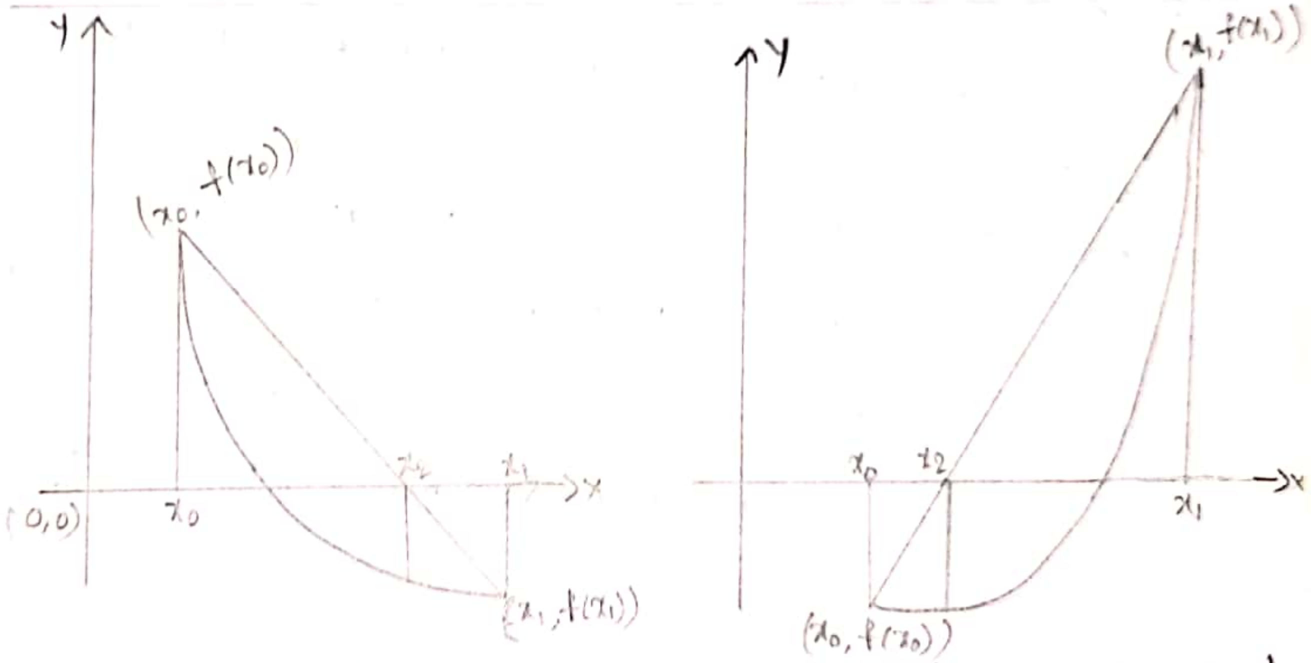
This implies that a root lies between x_0 and x_1

The curve $f(x)$ crosses x-axis only once at the point x_2 lying between the points x_0 and x_1.

Consider the point $A(x_0, f(x_0))$ and $B(x_1, f(x_1))$ on the graph and suppose they are connected by a straight line.

Suppose this line cuts x-axis at x_2 We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ and $f(x_2)$ are of opposite signs then the root lies between x_0 and x_2 and value x_1 is replaced by x_2

Otherwise the root lies between x_2 and x_1 and the value x_0 is replaced by x_2.



Another line is drawn by connecting the newly obtained pair of values. Again the point here the line cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired acuracy.

Note :- (i) The method requires two initial approximations to the root

(ii) The method always converges to the root (8)

(iii) The cost of the method is one evaluation of $f(x)$ per iteration.

(iv) If the root lies initially in (x_0, x_1), then one of the end points is fixed for all iterations In figure 1., the end point x_0 is fixed while in figure 2, the end point x_1 is fixed. Then the method is of the form. $x_{i+1} = \frac{x_0 f_i - x_i f_0}{f_i - f_0}$ $i = 1, 2, 3, \ldots$

This is the disadvantage of the method. However, it can be speeded up by inserting a bisection iteration after few iterations of the method of false position.

(v). The method has linear convergence.

Demerits of Regula Falsi Method :—

(i) Always one has to find the interval $[a,b]$ which brackets the zero of $f(x) = 0$. more importantly $f(a) * f(b)$ must be less than zero.

(ii) Although the length of the interval is getting smaller in each iteration, it is possible that it may not go to zero. If the graph $y = f(x)$ is Concave near the root 'c'. one of the endpoints becomes fixed and the other one marches towards the root.

→ Find a real root for $e^x \sin x = 1$ using Regula Falsi method.

Sol. - Given that the equation is $e^x \sin x - 1 = 0$

Let $f(x) = e^x \sin x - 1$

The sin function values measured in radians

$x = 0.5 \quad f(0.5) = e^{0.5}\sin(0.5) - 1 = 0.7904390832 - 1 = -0.20956 < 0$

$x = 0.6 \quad f(0.6) = e^{0.6}\sin(0.6) - 1 = 1.028845888 - 1 = 0.02885 > 0$

$x = 0.7 \quad f(0.7) = e^{0.7}\sin(0.7) - 1 = 1.297295112 - 1 = 0.297295 > 0$

We note that $f(0.5)$ and $f(0.6)$ are of opposite signs

∴ The given equation does have a real root between 0.5 and 0.6.

By Regula Falsi method

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

First Approximation :-

$$i = 1. \quad x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Let $x_0 = 0.5 \quad f(x_0) = f(0.5) = -0.20956$

$x_1 = 0.6 \quad f(x_1) = f(0.6) = 0.02885$

$$x_2 = \frac{0.5(0.02885) - 0.6(-0.20956)}{0.02885 - (-0.20956)}$$

$$x_2 = \frac{0.014425 - 0.125736}{0.23841} = 0.5879$$

$f(x_2) = f(0.5879) = e^{0.5879}\sin(0.5879) - 1 = 0.99842[illegible] - 1 = -0.00158 < 0$

We observe that $f(0.5879)$ and $f(0.6)$ are of opposite signs

∴ The root lies between 0.5879 and 0.6

Second Approximation :-

$i=2$, $\quad x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$x_1 = 0.5879$, $\quad f(x_1) = f(0.5879) = -0.00158$

$x_2 = 0.6 \quad f(x_2) = f(0.6) = 0.02885$

$$x_3 = \frac{0.5879(0.02885) - 0.6(-0.00158)}{0.02885 - (-0.00158)}$$

$$x_3 = \frac{0.016960915 + 0.000948}{0.03043} = 0.5885$$

$f(x_3) = f(0.5885) = e^{0.5885} \sin(0.5885) - 1 = -0.0000818 < 0$.

We observe that $f(0.5885)$ and $f(0.6)$ are of opposite signs.

$\therefore$ The root lies between 0.5885 and 0.6.

Third Approximation :—

$i=3$, $\quad x_4 = \dfrac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$

$x_2 = 0.5885 \quad f(x_2) = f(0.5885) = -0.0000818$

$x_3 = 0.6 \quad f(x_3) = f(0.6) = 0.02885$

$$x_4 = \frac{0.5885(0.02885) - 0.6(0.0000818)}{0.02885 - (-0.0000818)}$$

$$x_4 = \frac{0.016978225 + 0.00004908}{0.0289318} = 0.5885.$$

We observe that second and third approximations are approximately equal.

$\therefore$ An approximate root of the given equation is 0.5885.

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mated values of a repeated root.

(ii) Most severe limitation in the use of this method is the requirement that $f'(x) \neq 0$ in the neighbourhood of the root α. Even a moderate value of $f'(x_0)$ may more than sampled by a large value of either $f(x_0)$ or $f'(x_0)$ to produce a value x that will result in a sequence that convergence to a root that we are not interested.

(iii) Since two function evaluations are required in each iteration Newton Raphson method requires more computing time.

Merits and demerits of Newton Raphson Method :-

Merits.

(i) In this method convergence is quite fast provided the starting value is close to the desired root.

(ii) If the root is simple, the convergence is quadratic.

(iii) The accuracy of Newtons method for the function $f(x)$ which possess continues first and second derivatives can be estimated. If $M = \max|f''(x)|$ and $m = \min|f''(x)|$ in an interval that contains the root α and the estimator x_1 and x_2 then.

$$|x_2 - \alpha| \leq (x-\alpha)^2 \frac{M}{m}$$

Thus the error decreases if $\left|(x_1-\alpha)^2 \frac{M}{m}\right| < 1$

(iv) Newton Raphson iteration is a single point iteration.

(v) This method can used for solving both algebraic and transcen-dental equations. It can also be used when the roots are complex

(vi) The convergence of Newton Raphson method is of quadratic convergence. This method converges more rapidly than other methods.

(vii) This is a powerful and elegent method to find the root of an equation. This method is used to improve the results obtained by the previous methods.

Demerits :-

(i) In deriving the formula for this method, it is assumed that α is a not a repeated root of $f(x)=0$. In this case the convergence of the iteration is not guaranteed. Thus the Newton Raphson method is not applicable to find the approxi-

Merits and demerits of Newton Raphson Method :—

Merits :-

1. In this method convergence is quite fast provided the starting value is close to the desired root.
2. If the root is simple, the convergence is quadratic.
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4. This method can be used for solving both algebraic and transcendental equations. It can also be used when the roots are complex.

Demerits :—

1. In deriving the formula for this method, it is assumed that α is a not a repeated root of $f(x)=0$. In this case the convergence of the iteration is not guarented. Thus the Newton Raphson method is not applicable to find the approximated values of a repeated root.
2. Most severe limitation in the use of this method is the requirement that $f'(x) \neq 0$ in the neighbourhood of the root α.

Even a moderate value of $f'(x_0)$ may more than sampled by a large value of either $f(x_0)$ or $f'(x_0)$ to produce a value x that will result in a sequence the converges to a root that we are not interested.

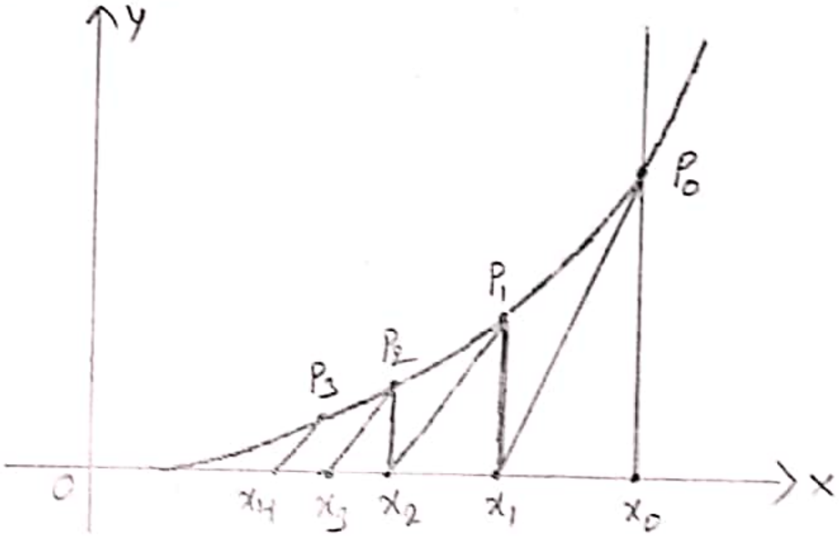
Geometrical significance :-

The Newton Raphson method starts with an initial approximation say x_0. Then a tangent is drawn from the corresponding point $f(x_0)$ on the curve $y = f(x)$

Let this tangent cuts the x-axis at a point say x_1.

Which will be a better approximation of the root.

Now calculate $f(x_1)$ and draw another tangent at the point $f(x_1)$ on the curve so that it cuts the x-axis at the point say x_2. The value of x_2 gives still better approximation and the process can be continued till the desired accuracy has been achieved.



Note :- (i) The method required one initial approximation.

(ii) The cost of the method is one evaluation of $f(x)$ and one evaluation of $f'(x)$ per iteration.

(iii) The method may fail if the initial approximation x_0 is far away from the root.

(iv) The method has second order convergence.

(v) This method fails if $f'(x) = 0$.

(vi) Newton's formula converges if $|f(x)f''(x)| < |f'(x)|^2$

Newton Raphson Method (Method of Tangents) :—

The Newton Raphson method is more advanced method in finding the root of the equation $f(x)=0$. It is used to improve the result obtained by Bisection or Regula falsi method.

Let $f(x)=0$ be the given equation.

Let x_0 be the approximate root of $f(x)=0$.

If x_1 is the exact root of the equation then $f(x_1)=0$

$$f(x_1) = f(x_0+(x_1-x_0)) = 0$$

By Taylor Series.

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

$$f(x_1) = f(x_0) + (x_1-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots = 0.$$

Suppose $x_1 - x_0$ is very small then the higher powers can be neglected.

$$\therefore f(x_0) + (x_1-x_0) f'(x_0) = 0$$

$$(x_1-x_0) f'(x_0) = -f(x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Generally $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

This is the Newton Raphson formula

(10) Find the root of the equation $x^3 - 4x^2 + x - 10 = 0$ in $(4, 5)$ using Method of chords.

(11) Find an approximate root of $xe^x = 3$ using Method of false position.

12) Find the root of $x^3 - 9x + 1 = 0$ using Regula Falsi method.

13) Explain Regula Falsi Method with geometrical interpretation.

14 Prove that the order of convergence of Regula Falsi method is 1.

REGULA FALSI METHOD.

(1) By using the Method of False position, find an approximate root of the equation $x^3-x-10=0$ that lies between 1.8 and 2. carry out three approximations. Ans: 1.8555.

(2) Using the method of false position, find the fifth root of 10 correct to four significant figures, assuming that the root lies between 1 and 2.

Ans:-

(3) By using the method of false position, find the root, correct to three decimal places of the equation $x\log_{10}x = 1.2$ that lies b/n 2 and 3.

Ans:-

(4) Using the Regula Falsi method, find the root, correct to three significant figures of the equation $xe^x = 2$ that lies between 0 and 1.

Ans: 0.853.

(5) Using the Method of False position find a real root (correct to three decimal places) of the equation $\cos x = 3x-1$ that lies b/n 0.5 and 1. Here x is in radians. Ans:- 0.607.

(6) Using the Regula Falsi Method, find the root, correct to four decimal places of the equation $xe^x = \cos x$ that lies between 0.4 and 0.6 (Here x is in radians) Ans:- 0.5177.

(7) Verify that the equation $\tan x + \tanh x = 0$ where x is in radians, has a root between 2 and 3. Find the 3rd approximation of this root by Regula Falsi Method. Ans: 2.3993.

(8) Using the Regula Falsi Method, find the negative root of $x^3-4x+9=0$.

(9) By using the method of false position, find the fourth roots of 12 and correct to three decimal places. Ans:- 1.861

→ Find the real root of the equation $x^3 - 2x - 5 = 0$ using Regula Falsi Method.

Sol:- Given that the equation is $f(x) = x^3 - 2x - 5 = 0$.

Let $f(x) = x^3 - 2x - 5$.

$x = 1 \quad f(1) = 1 - 2 - 5 = -6 < 0$

$x = 2 \quad f(2) = 8 - 4 - 5 = -1 < 0$

$x = 3 \quad f(3) = 27 - 6 - 5 = 16 > 0.$

We observe that $f(2)$ and $f(3)$ are of opposite signs.

∴ The given equation does have a real root between 2 and 3.

By Regula Falsi Method.

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}.$$

First Approximation :—

$$i = 1, \quad x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

Let $x_0 = 2 \quad f(x_0) = f(2) = -1.$

$x_1 = 3 \quad f(x_1) = f(3) = 16.$

$$x_2 = \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.05882.$$

$$f(x_2) = f(2.05882) = (2.05882)^3 - 2(2.05882) - 5.$$

$$= -0.39084 < 0$$

We observe that $f(2.05882)$ and $f(3)$ are of opposite signs.

∴ The root lies between 2.05882 and 3.

Second Approximation :—

$$i=2, \quad x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$x_1 = 2.05882 \qquad f(x_1) = -0.39084$

$x_2 = 3 \qquad f(x_2) = 16$.

$$x_3 = \frac{(2.05882).16 - 3(-0.39084)}{16 + 0.39084}$$

$$= \frac{34.11364}{16.39084} = 2.08126.$$

$$f(x_3) = f(2.08126) = (2.08126)^3 - 2(2.08126) - 5$$

$$f(x_3) = -0.14724 < 0.$$

We observe that $f(2.08126)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.08126 and 3.

Third Approximation :—

$$i=3 \qquad x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$x_2 = 2.08126 \quad f(x_2) = f(2.08126) = -0.14724$

$x_3 = 3 \qquad f(x_3) = f(3) = 16$

$$x_4 = \frac{(2.08126)16 - 3(-0.14724)}{16 + 0.14724}$$

$$= \frac{33.74188}{16.14724} = 2.08964$$

$$f(x_4) = f(2.08964) = (2.08964)^3 - 2(2.08964) - 5$$

$$= -0.05467 < 0.$$

We observe that $f(2.08964)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.08964 and 3.

Fourth Approximation :—

$i=4 \qquad x_5 = \dfrac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$

$x_3 = 2.08964 \qquad f(x_3) = -0.05467$

$x_4 = 3 \qquad f(x_4) = 16$.

$$x_5 = \frac{(2.08964).16 - 3(-0.05467)}{16 + 0.05467}$$

$$= \frac{33.59825}{16.05467} = 2.09274$$

$$f(x_5) = f(2.09274) = (2.09274)^3 - 2(2.09274) - 5$$

$$= -0.020198 < 0 .$$

We observe that $f(2.09274)$ and $f(3)$ are of opposite signs

$\therefore$ The root lies between 2.09274 and 3 .

Fifth Approximation :—

$i=5, \qquad x_6 = \dfrac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)}$

$x_4 = 2.09274 \qquad f(x_4) = -0.020198$

$x_5 = 3 \qquad f(x_5) = 16 > 0$.

$$x_6 = \frac{(2.09274)16 - 3(-0.020198)}{16 + 0.020198}$$

$$= \frac{33.544434}{16.020198} = 2.09388$$

$$f(x_6) = f(2.09388) = (2.09388)^3 - 2(2.09388) - 5 .$$

$$= -0.007491 < 0 .$$

We observe that $f(2.09388)$ and $f(3)$ are of opposite signs .

$\therefore$ The root lies between 2.09388 and 3 .

Sixth Approximation :-

$i=6 \qquad x_7 = \dfrac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)}$

$x_5 = 2.09888 \qquad f(x_5) = -0.007491$

$x_6 = 3 \qquad f(x_6) = 16$

$$x_7 = \frac{(2.09888)16 - 3(-0.007491)}{16 + 0.007491}$$

$$= \frac{33.524553}{16.007491} = 2.0943$$

$f(x_7) = f(2.0943) = (2.0943)^3 - 2(2.0943) - 5 = -0.002806$

We observe that $f(2.0943)$ and $f(3)$ are of opposite signs.

$\therefore$ The root lies between 2.0943 and 3.

Seventh Approximation :—

$i=7, \qquad x_8 = \dfrac{x_6 f(x_7) - f(x_6) x_7}{f(x_7) - f(x_6)}$

$x_6 = 2.0943 \qquad f(x_6) = -0.002806$

$x_7 = 3 \qquad f(x_7) = 16$

$$x_8 = \frac{(2.0943)16 - 3(-0.002806)}{16 + 0.002806}$$

$$x_8 = \frac{33.517218}{16.002806} = 2.0945$$

6th and 7th approximations are approximately equal.

$\therefore$ An approximate root of the given equation is 2.0945.

→ obtain Newton Raphson formula to find the square root of N and hence deduce the value of $\sqrt{8}$.

Sol:- Let $x = \sqrt{N}$

$x^2 = N$

$x^2 - N = 0$

Let $f(x) = x^2 - N$

$f'(x) = 2x$

The Newton Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}.$$

Which is the Newton Raphson formula for find a square root of N.

To find square root of $N = 8$:-

Let $f(x) = x^2 - 8$

$x = 2$ $f(2) = 4 - 8 = -4 < 0$

$x = 3$ $f(3) = 9 - 8 = 1 > 0.$

∴ The root lies between 2 and 3.5

Let the initial approximation $x_0 = 2.5$

1st Approximation :-

$n = 0$, $x_1 = \frac{x_0^2 + 8}{2x_0}$

$$x_1 = \frac{(2.5)^2 + 8}{2(2.5)} = 2.85.$$

2nd Approximation :-

$$n=1, \quad x_2 = \frac{x_1^2+8}{2x_1}$$

$$x_2 = \frac{(2.85)^2+8}{2(2.85)}$$

$$x_2 = 2.8285$$

3rd Approximation :-

$$n=2, \quad x_3 = \frac{x_2^2+8}{2x_2}$$

$$x_3 = \frac{(2.8285)^2+8}{2(2.8285)}$$

$$x_3 = 2.8284 .$$

2nd and 3rd approximations are approximately equal.

$\therefore$ square root of 8 is equals to 2.8284 approximately

→ Obtain Newton Raphson formula to find the fifth root of N. and hence deduce the value of $\sqrt[5]{5}$.

Sol:- Let $x = \sqrt[5]{N}$.

$$x = N^{1/5}$$

$$x^5 = N \implies x^5 - N = 0.$$

Let $f(x) = x^5 - N$.

$$f'(x) = 5x^4 .$$

The Newton Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

$$x_{n+1} = x_n - \frac{x_n^5 - N}{5x_n^4}$$

$$x_{n+1} = \frac{4x_n^5 + N}{5x_n^4}$$

Which is the Newton Raphson formula for finding the fifth root of N.

To find fifth root of N = 5 :-

Let $f(x) = x^5 - N$.

$f(x) = x^5 - 5$.

$x = 1 \quad f(1) = 1 - 5 = -4 < 0$;

$x = 1.2 \quad f(1.2) = (1.2)^5 - 5 = -2.51168 < 0$.

$x = 1.5 \quad f(1.5) = (1.5)^5 - 5 = 2.59375 > 0$.

$\therefore$ The root lies between 1.2 and 1.5

Let the initial approximation $x_0 = 1.3$.

1st Approximation :-

$$n=0, \quad x_1 = \frac{4x_0^5 + 5}{5x_0^4}$$

$$x_1 = \frac{4(1.3)^5 + 5}{5(1.3)^4} = 1.39013$$

2nd Approximation :—

$$n=1, \quad x_2 = \frac{4x_1^5 + 5}{5x_1^4}$$

$$x_2 = \frac{4(1.39013)^5 + 5}{5(1.39013)^4} = 1.37988.$$

3rd Approximation :—

$$n=2, \quad x_3 = \frac{4x_2^5 + 5}{5x_2^4}$$

$$x_3 = \frac{4(1.37988)^5 + 5}{5(1.37988)^4} = 1.37973$$

4th Approximation :—

$n=3, \quad x_4 = \dfrac{4x_3^5+5}{5x_3^4}$

$= \dfrac{4(1.37393)^5+5}{5(1.37393)^4}$

$= 1.37978$

3rd and 4th approximations are approximately equal.

∴ Fifth root of 5 is equal to 1.37978 approximately.

→ Obtain Newton Raphson formula to find the reciprocal of a number N and hence deduce the value of $\frac{1}{7}$.

Sol: Let $x = \dfrac{1}{N}$.

$N = \dfrac{1}{x} \implies \dfrac{1}{x} - N = 0.$

Let $f(x) = \dfrac{1}{x} - N.$

$f'(x) = \dfrac{-1}{x^2}.$

The Newton Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \dfrac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}$

$= x_n + \dfrac{\frac{1-Nx_n}{x_n}}{\frac{1}{x_n^2}}$

$= x_n + x_n - Nx_n^2.$

$x_{n+1} = 2x_n - Nx_n^2.$

Which is the Newton Raphson formula for finding the value of reciprocal of N.

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To find reciprocal of N = 7

Let $f(x) = \frac{1}{x} - 7$

$x = 0.1 \qquad f(0.1) = \frac{1}{0.1} - 7 = 3 > 0$

$x = 1 \qquad f(1) = 1 - 7 = -6 < 0$

$\therefore$ The root lies between 0.1 and 1

Let the initial approximation $x_0 = 0.1$

1st Approximation :—

$n = 0, \quad x_1 = 2x_0 - 7x_0^2$

$x_1 = 2(0.1) - 7(0.1)^2 = 0.13$

2nd Approximation :—

$n = 1, \quad x_2 = 2x_1 - 7x_1^2$

$x_2 = 2(0.13) - 7(0.13)^2 = 0.1417$

3rd Approximation :—

$n = 2, \quad x_3 = 2x_2 - 7x_2^2$

$x_3 = 2(0.1417) - 7(0.1417)^2 = 0.1428$

4th Approximation :—

$n = 3 \quad x_4 = 2x_3^2 - 7x_3^2$

$= 2(0.1428) - 7(0.1428)^2$

$= 0.14286.$

3rd and 4th approximations are approximately equal.

$\therefore$ Reciprocal of $7 = 0.14286$.

→ Find a real root of the equation $xe^x - \cos x = 0$ using Newton Raphson Method.

Sol:- Given that the equation $xe^x - \cos x = 0$

Let $f(x) = xe^x - \cos x$

The cos function values measured in radians

$x=0$, $f(0) = 0 - \cos 0 = -1 < 0$

$x=1$ $f(1) = e^1 - \cos 1 = 2.17 > 0$

∴ The root lies between 0 and 1.

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1st Approximation :—

$n=0$, $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x) = xe^x - \cos x$ $\quad f'(x) = (x+1)e^x + \sin x$.

Let the initial approximation $x_0 = 0.5$.

$f(x_0) = f(0.5) = (0.5)e^{0.5} - \cos(0.5) = -0.05322$

$f'(x_0) = f'(0.5) = (0.5+1)e^{0.5} + \sin(0.5) = 2.9525$.

$$x_1 = 0.5 - \frac{(-0.05322)}{2.9525}$$

$$x_1 = 0.5 + \frac{0.05322}{2.9525} = 0.518025$$

2nd Approximation :-

$n=1$ $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

At $x_1 = 0.518025$

$f(x_1) = f(0.518025) = (0.518025)e^{0.518025} - \cos(0.518025)$

$= 0.000814 4$.

$$f'(x_1) = f'(0.518025) = (0.518025+1)e^{0.518025} + \sin(0.518025)$$

$$= 3.04348734$$

$$x_2 = 0.518025 - \frac{0.0008144}{3.04348734}$$

$$= 0.518025 - 0.00026759$$

$$x_2 = 0.5178$$

<u>3rd Approximation</u>:

$$n=2, \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

At $x_2 = 0.5178$ $\quad f(x_2) = f(0.5178) = (0.5178)e^{0.5178} - \cos(0.5178)$

$$= 0.0001297$$

$$f'(x_2) = f'(0.5178) = (0.5178+1)e^{0.5178} + \sin(0.5178)$$

$$= 3.04234$$

$$x_3 = 0.5178 - \frac{0.0001297}{3.04234}$$

$$= 0.5178 - 0.00004263$$

$$x_3 = 0.51776$$

2nd and 3rd Approximations are approximately equal.

$\therefore$ The root of the equation is 0.51776.

NEWTON RAPHSON METHOD

(1) Using the Newton Raphson method find a real root, correct to four significant figures, of the equation $x^4 + 2x^3 - 16x + 5 = 0$ which lies near $x = 0$.

Ans:- 0.3265

(2) Apply Newton Raphson method to find an approximate root of the equation $x^3 - 3x - 5 = 0$ which lies near $x = 2$, correct five decimal places.

Ans- 2.27902.

(3) By Newton Raphson method find the real root of the equation $xe^x = 2$ correct to four significant figures.

Ans:- 0.8526.

(4) Using the Newton Raphson method, find the real root, correct to three decimal places, of the equation $\cos x - xe^x = 0$ which lies near $x = 0.5$ (Here x is in radians) correct to four significant figures.

Ans:-

(5) Using the Newton Raphson method, find the root that lies near $x = 4.5$ of the equation $\tan x = x$ correct to four significant figures.

Ans:-

(6) Using the Newton Raphson method find the real root of the equation $x \sin x + \cos x = 0$ near $x = \pi$. upto four decimal places (Here x is in radians)

Ans:-

(7) By Newton Raphson method, find the real root of the equation $x^2 + x + \cos x = 0$ near 0.5. Correct to 4 decimal places (Here x is in radi-ans)

(8) By Newton Raphson method, find the real root of the equation $x^3 - 6x + 4 = 0$ near 0.75 correct to 4 decimal places.

(9) obtain Newton Raphson formula to find square root of N and hence deduce the square root of 7.

Ans:- $x_{n+1} = \frac{x_n^2 + N}{2x_n}$

(10) obtain Newton Raphson formula to find cube root of N and hence deduce the cube root of 17.

Ans:- $x_{n+1} = \frac{2x_n^3 + 12}{3x_n^2}$

(11) obtain Newton Raphson formula to find fifth root of N and hence deduce the fifth root of 5.

Ans:-

(12) obtain Newton Raphson formula to find k^{th} root of positive number N and hence deduce the 7^{th} root of 7.

Ans:- $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$.

(13) obtain Newton Raphson formula to find the reciprocal of N and hence deduce the value of $\frac{1}{31}$.

Ans:- $x_{n+1} = 2x_n - N x_n^2$

(14) obtain Newton Raphson formula to find the reciprocal of k^{th} root of N and hence deduce the value of $\frac{1}{\sqrt{12}}$.

Ans:- $x_{n+1} = \frac{x_n}{k}\left[k+1 - N x_n^k\right]$.

(15) Find an approximate positive root of the equation $e^x \sin x = 1$ using the Newton Raphson method (Here x, is in radians)

Ans:- 0.58853.

(16) Explain Newtons Raphson method with geometrical interpretation.

(17) Prove that the order of convergence of Newton Raphson method is 2.

Iteration Method :

Iteration is the process in which we perform the same procedure again and again.

In iterative method we can solve the problem by calculating the successive approximations to the solution with an initial guess.

Iterative methods are useful for the problems involving a large no. of variables.

Let $f(x) = 0$ be the given equation.

It can be expressed as $x = \phi(x)$.

Let x_0 be an approximate value of the desired root α.

We calculate $x_1 = \phi(x_0)$.

The successive approximations are given by

$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2)$$

$$\cdots$$

$$x_n = \phi(x_{n-1}).$$

We proceed in this way until the two successive approximations are approximately equal.

$x_n = \phi(x_{n-1})$ is called the iterative formula

This Method is convergent if $|\phi'(x)| < 1 \;\; \forall x \in I$.

Where I is the interval which contains a root of the equation.

1) By the fixed point iteration process, find the root correct to 3-decimal places of the equation $x = \cos x$ near $x = \frac{\pi}{4}$.

Sol:- Given that the equation is $x = \cos x$.

which is of the form $x = \phi(x)$

where $\phi(x) = \cos x$.

$\phi'(x) = \sin x$.

$|\phi'(x)| = |\sin x| < 1 \quad \forall x$

Since the root is required near $x = \frac{\pi}{4}$.

we take the initial approximation of the root as $x_0 = \frac{\pi}{4} = 0.78571$

The iteration formula is $x_n = \phi(x_{n-1})$.

$n=1, \quad x_1 = \phi(x_0) = \cos(0.78571) = 0.70689$.

$n=2, \quad x_2 = \phi(x_1) = \cos(0.70689) = 0.76039$

$n=3 \quad x_3 = \phi(x_2) = \cos(0.76039) = 0.72457$.

$n=4, \quad x_4 = \phi(x_3) = \cos(0.72457) = 0.74878$

$n=5, \quad x_5 = \phi(x_4) = \cos(0.74878) = 0.73252$

$n=6, \quad x_6 = \phi(x_5) = \cos(0.73252) = 0.74349$

$n=7 \quad x_7 = \phi(x_6) = \cos(0.74349) = 0.73611$

$n=8 \quad x_8 = \phi(x_7) = \cos(0.73611) = 0.74109$

$n=9 \quad x_9 = \phi(x_8) = \cos(0.74109) = 0.73773$.

$n=10 \quad x_{10} = \phi(x_9) = \cos(0.73773) = 0.739997$.

$n=11 \quad x_{11} = \phi(x_{10}) = \cos(0.739997) = 0.73847$.

$n=12 \quad x_{12} = \phi(x_{11}) = \cos(0.73847) = 0.73949$.

. . . .

12th and 13th approximations are approximately equal.

$\therefore$ The root of the equation is 0.739.

(2) Find the positive root of the equation $x^4-x-10=0$ by iteration

sol:- Given that the equation is $x^4-x-10=0$.

It can be written as $x=\phi(x)$ in many ways such as

$$x=x^4-10 \qquad x=\frac{10}{x^3-1} \qquad x=(x+10)^{\frac{1}{4}}$$

only $x=(x+10)^{1/4}$ satisfies the converge criteria $|\phi'(x)|<1$.

Let $f(x)=x^4-x-10$

$x=1$, $f(1)=-10<0$

$x=2$ $f(2)=16-2-10=4>0$.

$\therefore$ The root lies between 1 and 2.

The Iteration formula is $x_n=\phi(x_{n-1})$.

$$\phi(x)=(x+10)^{1/4}$$

$$\phi'(x)=\frac{1}{4}(x+10)^{\frac{-3}{4}}.$$

$$|\phi'(x)|<1 \quad \forall x\in[1,2].$$

choose $x_0=1.5$

$n=1$, $x_1=\phi(x_0)=(x_0+10)^{\frac{1}{4}}=(1.5+10)^{\frac{1}{4}}=1.8415$.

$n=2$ $x_2=\phi(x_1)=(x_1+10)^{1/4}=(1.8415+10)^{1/4}=1.8550$.

$n=3$ $x_3=\phi(x_2)=(x_2+10)^{1/4}=(1.8550+10)^{1/4}=1.8556$.

$n=4$ $x_4=\phi(x_3)=(x_3+10)^{1/4}=(1.8556+10)^{1/4}=1.8556$.

x_3 and x_4 are approximately equal.

$\therefore$ The approximate root of the equation is 1.8556.

Geometrical Interpretation of iteration Method :—

The iteration method can be represented geometrically as follows.

Let $x_0, x_1, x_2, \ldots x_n$ denote the successive approximations to the root ξ

Then $x_1 = \phi(x_0)$

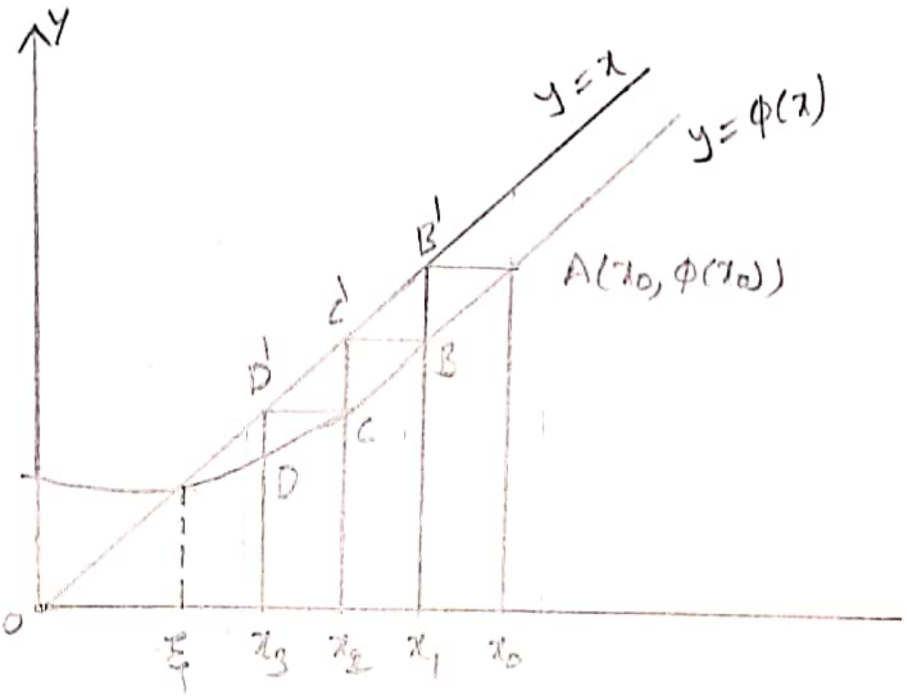
$x_2 = \phi(x_1)$

$x_3 = \phi(x_2)$ etc.

can be viewed as points by the following geometric constructions.

By sketching the line $y=x$ and the curve $y=\phi(x)$ the intersection of these two curves gives the exact root $x=\xi$. Since $|\phi'(x)|<1$, then in the neighbourhood of x_0 the inclination of the curve of $y=\phi(x)$ must be less than $\frac{\pi}{4}$ or 45°

In order to trace the convergence of the iteration method we first draw the ordinate $\phi(x_0)$ parallel to y-axis, which meets the curve $\phi(x)$ at point $A(x_0, \phi(x_0))$ then from point A draw a line parallel to x-axis which meets the line $y=x$ at the points $B'(x_1, \phi(x_0))$, it is noted here that this point B' is the geometric representation of the first iteration equation $x_1 = \phi(x_0)$.



Now draw a line from B' parallel to y-axis which meets the curve $\phi(x)$ at point $B(x_1, \phi(x_1))$ and again draw a line from B parallel to x-axis which ~~E~~ meets the line $y = x$ at c' whose abssicsa is x_2 which gives the second approximation., Continue this process of drawing the lines parallel to the co ordinate axes, we finally approaches to ξ.

ITERATION METHOD

(1) Using the Iteration method, find the real root of the equation $x^3-x-1=0$ that is near $x=1$. correct to four decimal places.

Ans:- 1.3247.

(2) By the fixed point iteration method, find the root of the equation $x^3-2x-5=0$ that is near $x=2$. correct to 5 significant figures.

Ans:- 2.0946.

(3) By the iteration process find the root, correct to 5 decimal places, of the equation $x = \cos x$ near $x = \frac{\pi}{4}$.

Ans: - 0.73899.

(4) Using the iteration method, find the root of the equation $3x = 1+\cos x$ in the interval $(0, 1)$ Here x is in radians.

Ans:- 0.60713.

(5) Evaluate $\sqrt{12}$ and $\frac{1}{\sqrt{12}}$ by using the iteration method.

Ans:- 3.464285, 0.28869.

(6) Solve $x = 1+\tan^{-1}x$ by using the iteration method.

Ans:- $x = 2.1323$.

(7) Solve $x^3 = 2x+5$ for a positive by iteration method.

Ans:-

(8) Using the method of iteration find a positive root between 0 and 1 of the equation $xe^x = 1$.

Ans:-

9) Solve $x^2-2x+1=0$ using iteration method.

10 Solve $\sin x - 2x + 1 = 0$ using iteration method by taking $x_0 = 1.5$

11) Solve $x^4 - 12x + 7 = 0$ using iteration method.

12) Solve $x^3 + x^2 - 1 = 0$ for a positive root by iteration method.

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Ramanujan's Method :-

Srinivasa Ramanujan described an iterative procedure to determine the smallest root of the equation $f(x)=0$. where $f(x)$ is of the form $f(x) = 1-(a_1x+a_2x^2+a_3x^3+\dots)$

To find the smallest root of $f(x)=0$, we consider $f(x)$ in the form

$$f(x) = 1-(a_1x+a_2x^2+a_3x^3+\dots) \quad \text{--- } (1)$$

and then write.

$$\left[1-(a_1x+a_2x^2+a_3x^3+\dots)\right]^{-1} = b_1+b_2x+b_3x^2+\dots$$

$$1+(a_1x+a_2x^2+a_3x^3+\dots)+(a_1x+a_2x^2+a_3x^3+\dots)^2+\dots$$

$$= b_1+b_2x+b_3x^2+\dots \quad \text{--- } (2)$$

To find b_i, we equate coeff. of the like powers of x on both sides of (2), we then obtain.

$$b_1 = 1.$$

$$b_2 = a_1 = a_1b_1 \quad \text{since } b_1=1.$$

$$b_3 = a_2+a_1^2 = a_2b_1+a_1b_2 \quad \text{since } b_2=a_1$$

$$\dots\dots$$

$$b_k = a_1b_{k-1}+a_2b_{k-2}+\dots+a_{k-1}b_1$$

$$b_k = a_{k-1}b_1+a_{k-2}b_2+\dots+a_1b_{k-1}.$$

The ratios $\frac{b_{i-1}}{b_i}$ called the convergents, approach, in the limit, the smallest root of $f(x)=0$.

→ Find the smallest root of the equation $x^3-9x^2+26x-24=0$.

Sol:- Given that $f(x)=x^3-9x^2+26x-24=0$.

We have $f(x)=24-26x+9x^2-x^3$.

$f(x)=1-\frac{26}{24}x+\frac{9}{24}x^2-\frac{1}{24}x^3$.

$f(x)=1-\left(\frac{13}{12}x-\frac{3}{8}x^2+\frac{1}{24}x^3\right)$.

Which is of the form $f(x)=1-(a_1x+a_2x^2+a_3x^3+\dots)$

Here $a_1=\frac{13}{12}$ $a_2=-\frac{3}{8}$ $a_3=\frac{1}{24}$. $a_4=0,\ a_5=0\dots$

Now $b_1=1$.

$b_2=a_1b_1=\frac{13}{12}=1.0833$

$\therefore \frac{b_1}{b_2}=\frac{12}{13}=0.923$.

$b_3=a_1b_2+a_2b_1$.

$=\frac{13}{12}(1.0833)-\frac{3}{8}(1)=0.7986$.

$\frac{b_2}{b_3}=\frac{1.0833}{0.7986}=1.356$.

$b_4=a_1b_3+a_2b_2+a_3b_1$.

$=(1.0833)(0.7986)+\left(-\frac{3}{8}\right)(1.0833)+\frac{1}{24}(1)$

$b_4=0.5007$

$\frac{b_3}{b_4}=1.595$

$b_5=a_1b_4+a_2b_3+a_3b_2+a_4b_1$

$=1.0833(0.5007)+\left(-\frac{3}{8}\right)(0.7986)+\frac{1}{24}(0.0833)$

$b_5=0.2880$

$\therefore \frac{b_4}{b_5}=\frac{0.5007}{0.2880}=1.7382$

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1.$$

$$b_6 = 0.1575$$

$$\therefore \frac{b_5}{b_6} = 1.8286.$$

$$b_7 = a_1 b_6 + a_2 b_5 + a_3 b_4 + a_4 b_3 + a_5 b_2 + a_6 b_1$$

$$b_7 = 0.0835$$

$$\therefore \frac{b_6}{b_7} = 1.8862.$$

$$b_8 = a_1 b_7 + a_2 b_6 + a_3 b_5 + a_4 b_4 + a_5 b_3 + a_6 b_2 + a_7 b_1$$

$$b_8 = 0.0434.$$

$$\therefore \frac{b_7}{b_8} = 1.9240.$$

$$b_9 = a_1 b_8 + a_2 b_7 + a_3 b_6 + a_4 b_5 + a_5 b_4 + a_6 b_3 + a_7 b_2 + a_8 b_1$$

$$b_9 = 0.0223$$

$$\therefore \frac{b_8}{b_9} = 1.9462.$$

The roots of the given equation are 2, 3 and 4 and it can be seen that the successive convergents approach the value 2.

RAMANUJAN'S METHOD

(1) Explain Ramanujan's Method.

(2) Find the smallest root of $x^3 - 9x^2 + 26x - 24 = 0$.

(3) Find the root of $xe^x = 1$ using Ramanujan's Method.

(4) Find the root of $3x - \cos x - 1 = 0$.

(5) Find the root of $1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} \cdots = 0$.

(6) Solve $x + x^3 - 1 = 0$.

(7) Find the smallest root of $x + \sin x - 1 = 0$.

(8) Find the smallest root of $x^3 - 6x^2 + 11x - 6 = 0$.

ITERATIVE METHODS OF SOLUTION :-

The preceding methods of solving simultaneous linear equations are known as direct methods as they yield exact solutions. on the other hand an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy.

Simple iteration methods can be devised for systems in which the coefficients of the leading diagonal are large compared to others.

JACOBI'S ITERATION METHOD :-

Working Procedure :-

Consider the equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{——— (1)}$$

Where the coefficients of the diagonal elements, a_{11}, a_{22}, a_{33} are all not equal to zero. and large compared to the other coefficients. Systems of this type are known as diagonally dominant.

Step 1 :- We write the equations as.

$$\left.\begin{aligned} x_1 &= \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3] \\ x_2 &= \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3] \\ x_3 &= \frac{1}{a_{33}}[b_3 - a_{31}x_1 - a_{32}x_2] \end{aligned}\right\} \text{——— (2).}$$

Step 2 :- Suppose $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$ are the initial approximate values of x_1, x_2, x_3. which satisfy equations (2). Substituting these values into the right sides of equations (2), we obtain a system of first approximations of x_1, x_2, x_3 or first iterates, given by

$$\left.\begin{aligned} x_1^{(1)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right] \\ x_2^{(1)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right] \\ x_3^{(1)} &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right] \end{aligned}\right\} \text{——— (3)}$$

Step 3 :- Substituting $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$ for x_1, x_2, x_3 in the right sides of equations ②. we obtain the second iterates or system of second approximations of x_1, x_2, x_3 is given by.

$$\left.\begin{aligned} x_1^{(2)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}\right] \\ x_2^{(2)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(1)}\right] \\ x_3^{(2)} &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right] \end{aligned}\right\} \text{———} ④$$

Step 4 :- Proceeding like this we get successive Iterates.

The $(n+1)^{th}$ iterates are given by.

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)}\right]$$

$$x_2^{(n+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(n)} - a_{23}x_3^{(n)}\right]$$

$$x_3^{(n+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(n)} - a_{32}x_2^{(n)}\right]$$

The process of iteration is stopped when the desired order of approxi-mation is reached or two successive iterations are nearly the same. The final values of x_1, x_2, x_3 so obtained constitute an approximate solution of the system ①.

We can extend this method to n equations in n unknowns.

Note :- In this method, the process of iteration starts with some initial approximation to the solution, namely $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$. This initial solution is choosen as zero solution. However, if an initial approximation is known before hand, it can be used to start the Iteration.

Jacobi's Iteration Method :-

Let us consider the system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{—} (1)$$

Where the coefficients of the diagonal elements are all not equal to zero and large compared to the other coefficients. Systems of this type are known as diagonally dominant systems.

The solution to the above system is obtained by iteration method called Jacobi's Iteration method.

Working Procedure :-

Step 1 :- Write the system of equations

$$\left.\begin{aligned} x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \\ x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \\ x_3 &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \end{aligned}\right\} \text{—} (2)$$

Step 2 :- Suppose $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$ are the initial approximate values of x_1, x_2

x_3 which satisfy equations (2). Substituting these values into the right

sides of equations (2), we obtain a system of first approximations of

$x_1\ x_2\ x_3$ or first iterates given by

$$\left.\begin{aligned} x_1^{(1)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right] \\ x_2^{(1)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right] \\ x_3^{(1)} &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right] \end{aligned}\right\} \text{—} (3).$$

Step 3 :- Substituting $x_1^{(1)}\ x_2^{(1)}\ x_3^{(1)}$ for $x_1\ x_2\ x_3$ in the right sides of (2).

we obtain the second iterates given by

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}\right],\ x_2^{(2)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(1)}\right],\ x_3^{(2)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right] \text{—} (4).$$

Proceeding like this we get successive iterates.

Step 4 :- Proceeding like this we get successive iterates

The $(r+1)^{th}$ iterates are given by

$$x_1^{(r+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12} x_2^{(r)} - a_{13} x_3^{(r)}\right]$$

$$x_2^{(r+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21} x_1^{(r)} - a_{23} x_3^{(r)}\right]$$

$$x_3^{(r+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31} x_1^{(r)} - a_{32} x_2^{(r)}\right].$$

The process of iteration is stopped when the desired order of approxi-mation is reached or two successive iterations are nearly the same. The final values of x_1, x_2, x_3 so obtained constitute an approximate solution of the system (1) .

Note :- In this method, the process of iteration starts with some initial approximation to the solution, namely $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$. This initial solution is choosen as zero solution. However if an initial approximation is known before hand, it can be used to start the iteration.

→ Solve by Jacobi's iteration method, the equations

$20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=25$

Sol:- Given that $20x+y-2z=17$

$3x+20y-z=-18$

$2x-3y+20z=25$

We observe that the given system is diagonally dominant.

We rewrite the given equations in the form

$$x=\frac{1}{20}(17-y+2z)$$

$$y=\frac{1}{20}(-18-3x+z)$$

$$z=\frac{1}{20}(25-2x+3y)$$

Jacobi's $(K+1)^{th}$ iterates are given by

$$x^{(k+1)}=\frac{1}{20}\left[17-y^{(K)}+2z^{(K)}\right]$$

$$y^{(k+1)}=\frac{1}{20}\left[-18-3x^{(K)}+z^{(K)}\right]$$

$$z^{(k+1)}=\frac{1}{20}\left[25-2x^{(K)}+3y^{(K)}\right].$$

We start from an approximation $x^{(0)}=y^{(0)}=z^{(0)}=0$.

First Approximation :—

$K=0$, $$x^{(1)}=\frac{1}{20}\left[17-y^{(0)}+2z^{(0)}\right]=\frac{17}{20}=0.85$$

$$y^{(1)}=\frac{1}{20}\left[-18-3x^{(0)}+z^{(0)}\right]=\frac{-18}{20}=-0.9$$

$$z^{(1)}=\frac{1}{20}\left[25-2x^{(0)}+3y^{(0)}\right]=\frac{25}{20}=1.25$$

Second Approximation :—

$K=1$, $$x^{(2)}=\frac{1}{20}\left[17-y^{(1)}+2z^{(1)}\right]=\frac{1}{20}\left[17+0.9+2.5\right]=1.02$$

$$y^{(2)}=\frac{1}{20}\left[-18-3x^{(1)}+z^{(1)}\right]=\frac{1}{20}\left[-18-3(0.85)+1.25\right]=-0.965$$

$$z^{(2)}=\frac{1}{20}\left[25-2x^{(1)}+3y^{(1)}\right]=\frac{1}{20}\left[25-2(0.85)+3(-0.9)\right]=1.03$$

Third Approximation :—

$K=2$ $\quad x^{(3)} = \frac{1}{20}\left[17 - y^{(2)} + 2z^{(2)}\right] = \frac{1}{20}\left[17-(-0.965)+2(1.03)\right] = 1.00125$

$$x^{(3)} = 1.00125$$

$$y^{(3)} = \frac{1}{20}\left[-18 - 3x^{(2)} + z^{(2)}\right] = \frac{1}{20}\left[-18 - 3(1.02) + (1.03)\right] = -1.0015$$

$$z^{(3)} = \frac{1}{20}\left[25 - 2x^{(2)} + 3y^{(2)}\right] = \frac{1}{20}\left[25 - 2(1.02) + 3(-0.965)\right] = 1.00325$$

Fourth Approximation :—

$K=3$ $\quad x^{(4)} = \frac{1}{20}\left[17 - y^{(3)} + 2z^{(3)}\right] = \frac{1}{20}\left[17-(-1.0015)+2(1.00325)\right] = 1.0004$

$$y^{(4)} = \frac{1}{20}\left[-18 - 3x^{(3)} + z^{(3)}\right] = \frac{1}{20}\left[-18 - 3(1.00125) + 1.00325\right]$$

$$y^{(4)} = -1.000025$$

$$z^{(4)} = \frac{1}{20}\left[25 - 2x^{(3)} + 3y^{(3)}\right] = \frac{1}{20}\left[25 - 2(1.00125) + 3(-1.0015)\right]$$

$$z^{(4)} = 0.99965$$

Fifth Approximation :—

$K=4,$ $\quad x^{(5)} = \frac{1}{20}\left[17 - y^{(4)} + 2z^{(4)}\right] = \frac{1}{20}\left[17-(-1.000025)+2(0.99965)\right]$

$$x^{(5)} = 0.99997$$

$$y^{(5)} = \frac{1}{20}\left[-18 - 3x^{(4)} + z^{(4)}\right] = \frac{1}{20}\left[-18 - 3(1.0004) + 0.99965\right]$$

$$y^{(5)} = -1.00008$$

$$z^{(5)} = \frac{1}{20}\left[25 - 2x^{(4)} + 3y^{(4)}\right] = \frac{1}{20}\left[25 - 2(1.0004) + 3(-1.000025)\right]$$

$$z^{(5)} = 0.99995625$$

The values in the 4th and 5th approximations being practically the same, we can stop.

Hence the solution is $x=1$ $\quad y=-1$ $\quad z=1$.

→ Solve by Jacobi's method, the equations $5x - y + z = 10$, $2x + 4y = 12$, $x + y + 5z = -1$ start with the solution $(2, 3, 0)$ 1

Sol:- Given that $5x + y + z = 10$

$2x + 4y = 12$

$x + y + 5z = -1$

We observe that given system is diagonally dominant.

We rewrite the system of equations as

$$x = \frac{1}{5}(10 + y - z)$$

$$y = \frac{1}{2}(6 - x)$$

$$z = \frac{1}{5}(-1 - x - y)$$

The $(n+1)^{th}$ Jacobi's iterates are given by.

$$x^{(n+1)} = \frac{1}{5}[10 + y^{(n)} - z^{(n)}]$$

$$y^{(n+1)} = \frac{1}{2}[6 - x^{(n)}] \quad \text{—— ①}$$

$$z^{(n+1)} = \frac{1}{5}[-1 - x^{(n)} - y^{(n)}]$$

An initial values of x, y, z are given by $x^{(0)} = 2$, $y^{(0)} = 3$, $z^{(0)} = 0$.

First iteration :—

put $n = 0$ in ①, we get

$$x^{(1)} = \frac{1}{5}[10 + y^{(0)} - z^{(0)}] = \frac{1}{5}[10 + 3 - 0] = 2.6$$

$$y^{(1)} = \frac{1}{2}[6 - x^{(0)}] = \frac{1}{2}[6 - 2] = 2$$

$$z^{(1)} = \frac{1}{5}[-1 - x^{(0)} - y^{(0)}] = \frac{1}{5}[-1 - 2 - 3] = -1.2$$

Second iteration :—

Put $n = 1$ in ①, we get

$$x^{(2)} = \frac{1}{5}[10 + y^{(1)} - z^{(1)}] = \frac{1}{5}[10 + 2 + 1.2] = 2.64$$

$$y^{(2)} = \frac{1}{2}[6 - x^{(1)}] = \frac{1}{2}[6 - 2.6] = 1.7$$

$$z^{(2)} = \frac{1}{5}[-1 - x^{(1)} - y^{(1)}] = \frac{1}{5}[-1 - 2.6 - 2] = -1.12$$

Third iteration :-

Put $n=2$ in (1), we get

$$x^{(3)} = \frac{1}{5}\left[10 + y^{(2)} - z^{(2)}\right] = \frac{1}{5}\left[10 + 1.7 + 1.12\right] = 2.564$$

$$y^{(3)} = \frac{1}{2}\left[6 - x^{(2)}\right] = \frac{1}{2}\left[6 - 2.64\right] = 1.68$$

$$z^{(3)} = \frac{1}{5}\left[-1 - x^{(2)} - y^{(2)}\right] = \frac{1}{5}\left[-1 - 2.64 - 1.7\right] = -1.068$$

Fourth iteration :-

Put $n=3$ in (1), we get

$$x^{(4)} = \frac{1}{5}\left[10 + y^{(3)} - z^{(3)}\right] = \frac{1}{5}\left[10 + 1.68 + 1.068\right] = 2.5496$$

$$y^{(4)} = \frac{1}{2}\left[6 - x^{(3)}\right] = \frac{1}{2}\left[6 - 2.564\right] = 1.718$$

$$z^{(4)} = \frac{1}{5}\left[-1 - x^{(3)} - y^{(3)}\right] = \frac{1}{5}\left[-1 - 2.564 - 1.68\right] = -1.0488$$

Fifth iteration :-

Put $n=4$ in (1), we get

$$x^{(5)} = \frac{1}{5}\left[10 + y^{(4)} - z^{(4)}\right] = \frac{1}{5}\left[10 + 1.718 + 1.0488\right] = 2.55336$$

$$y^{(5)} = \frac{1}{2}\left[6 - x^{(4)}\right] = \frac{1}{2}\left[6 - 2.5496\right] = 1.7252$$

$$z^{(5)} = \frac{1}{5}\left[-1 - x^{(4)} - y^{(4)}\right] = \frac{1}{5}\left[-1 - 2.5496 - 1.718\right] = -1.05352$$

Sixth iteration :-

Put $n=5$ in (1), we get

$$x^{(6)} = \frac{1}{5}\left[10 + y^{(5)} - z^{(5)}\right] = \frac{1}{5}\left[10 + 1.7252 + 1.05352\right] = 2.555744$$

$$y^{(6)} = \frac{1}{2}\left[6 - x^{(5)}\right] = \frac{1}{2}\left[6 - 2.55336\right] = 1.72332$$

$$z^{(6)} = \frac{1}{5}\left[-1 - x^{(5)} - y^{(5)}\right] = \frac{1}{5}\left[-1 - 2.55336 - 1.7252\right] = -1.055712$$

Seventh iteration :-

Put $n=6$ in (1), we get

$$x^{(7)} = \frac{1}{5}\left[10 + y^{(6)} - z^{(6)}\right] = \frac{1}{5}\left[10 + 1.72332 + 1.055712\right] = 2.5558064$$

$$y^{(7)} = \frac{1}{2}\left[6 - x^{(6)}\right] = \frac{1}{2}\left[6 - 2.555744\right] = 1.722128$$

$$z^{(7)} = \frac{1}{5}\left[-1 - x^{(6)} - y^{(6)}\right] = \frac{1}{5}\left[-1 - 2.555744 - 1.72332\right] = -1.0558128$$

We observe that the solutions in the 6th and 7th iterations are approximately equal.

$\therefore$ The solution of the system is $x = 2.5558$, $y = 1.7221$, $z = -1.0558$.

Gauss Seidel Iteration Method :-

This is a modification of Gauss Jacobi's method

We will consider the system of equations.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \text{———} (1)$$

Where the diagonal coefficients are not zero and are large compared to other coefficients. such a system is called a diagonally dominant system.

Step 1 :- The system of equations (1) may be written as

$$\left.\begin{aligned} x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \\ x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \\ x_3 &= \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \end{aligned}\right\} \text{———} (2)$$

Step 2 :- Let the initial approximate solution be $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$ substituting $x_2^{(0)}$, $x_3^{(0)}$ in the first equation of (2), we get

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right] \text{———} (3)(a).$$

This is taken as the first approximation of x_1.

Substituting $x_1^{(1)}$ for x and $x_3^{(0)}$ for $x_3^{(0)}$ in the second equation of (2).

we get $$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right] \text{———} (3)(b).$$

This is taken as the first approximation of x_2.

Next, Substituting $x_1^{(1)}$ for x_1 and $x_2^{(1)}$ for x_2 in the last equation of (2), we get. $$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right] \text{———} (3)(c).$$

This is taken as the first approximation of x_3.

The values obtained in 3(a), 3(b), 3(c) constitute the first iterates of the solution.

Step 3:- Proceeding in the same way, we get successive iterates.

The $(k+1)^{th}$ iterates are given by

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right].$$

The iteration process is stopped when the desired order of approximation is reached or two successive iterations are nearly the same.

The final values of x_1 x_2 x_3 so obtained constitute an approximate solution of the system ②.

Apply Gauss Seidel iteration method to solve the equations $20x+y-2z=17$

$3x+20y-z=-18$, $2x-3y+20z=25$

Sol:- Given that the equations $20x+y-2z=17$, $3x+20y-z=-18$ $2x-3y+20z=25$

We observe that given system is diagonally dominant

We rewrite the system of equations as

$$x=\frac{1}{20}(17-y+2z) \quad y=\frac{1}{20}(-18-3x+z) \quad z=\frac{1}{20}(25-2x+3y)$$

The $(k+1)^{th}$ iterates of Gauss seidel iteration method is given by

$$x^{(k+1)}=\frac{1}{20}\left[17-y^{(k)}+2z^{(k)}\right]$$

$$y^{(k+1)}=\frac{1}{20}\left[-18-3x^{(k+1)}+2z^{(k)}\right].$$

$$z^{(k+1)}=\frac{1}{20}\left[25-2x^{(k+1)}+3y^{(k+1)}\right]$$

First Approximation :—

We start iteration by taking $y^{(0)}=0$ $z^{(0)}=0$.

$k=0$, $\quad x^{(1)}=\frac{1}{20}\left[17-y^{(0)}+2z^{(0)}\right]=\frac{1}{20}(17-0+0)=0.85.$

$$y^{(1)}=\frac{1}{20}\left[-18-3x^{(1)}+2z^{(0)}\right]=\frac{1}{20}\left[-18-3(0.85)+2(0)\right].$$

$$y^{(1)}=-1.0275$$

$$z^{(1)}=\frac{1}{20}\left[25-2x^{(1)}+3y^{(1)}\right]=\frac{1}{20}\left[25-2(0.85)+3(-1.0275)\right]$$

$$z^{(1)}=1.010875.$$

Second Approximation :—

$k=1$, $\quad x^{(2)}=\frac{1}{20}\left[17-y^{(1)}+2z^{(1)}\right]=\frac{1}{20}\left[17-(-1.0275)+2(1.010875)\right]$

$$x^{(2)}=1.0024625$$

$$y^{(2)}=\frac{1}{20}\left[-18-3x^{(2)}+2z^{(1)}\right]=\frac{1}{20}\left[-18-3(1.0024625)+2(1.010875)\right]$$

$$y^{(2)}=-0.999826$$

$$z^{(2)} = \frac{1}{20}\left[25 - 2x^{(2)} + 3y^{(2)}\right]$$

$$= \frac{1}{20}\left[25 - 2(1.0024625) + 3(-0.999826)\right] = 0.99978$$

Third Approximation :—

$k = 2,$ $$x^{(3)} = \frac{1}{20}\left[17 - y^{(2)} + 2z^{(2)}\right]$$

$$= \frac{1}{20}\left[17 - (-0.999826) + 2(0.99978)\right]$$

$$x^{(3)} = 0.9999693$$

$$y^{(3)} = \frac{1}{20}\left[-18 - 3x^{(3)} + z^{(2)}\right]$$

$$= \frac{1}{20}\left[-18 - 3(0.9999693) + 0.99978\right]$$

$$y^{(3)} = -1.000006$$

$$z^{(3)} = \frac{1}{20}\left[25 - 2x^{(3)} + 3y^{(3)}\right]$$

$$= \frac{1}{20}\left[25 - 2(0.9999693) + 3(-1.000006)\right]$$

$$= 1.000002$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is $x = 1$ $y = -1$ $z = 1$.

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Solve by Gauss - Seidal method, the equations $5x-y+z=10$, $2x+4y=12$, $x+y+5z=-1$. Start with the solution $(2,3,0)$.

Sol:- Given that $5x-y+z=10$

$2x+4y=12$

$x+y+5z=-1$

We observe that given system is diagonally dominant.

We rewrite the system of equations as

$$x=\frac{1}{5}(10+y-z)$$

$$y=\frac{1}{2}(6-x)$$

$$z=\frac{1}{5}(-1-x-y)$$

The $(n+1)^{th}$ Gauss Seidal iterates are given by

$$\left.\begin{aligned} x^{(n+1)} &= \frac{1}{5}\left[10+y^{(n)}-z^{(n)}\right]. \\ y^{(n+1)} &= \frac{1}{2}\left[6-x^{(n+1)}\right]. \\ z^{(n+1)} &= \frac{1}{5}\left[-1-x^{(n+1)}-y^{(n+1)}\right]. \end{aligned}\right\} \quad \text{——— (1)}$$

An initial values of x, y, z are given by $x^{(0)}=2$, $y^{(0)}=3$, $z^{(0)}=0$.

First iteration :—

Put $n=0$ in (1), We get

$$x^{(1)}=\frac{1}{5}\left[10+y^{(0)}-z^{(0)}\right]$$

$$=\frac{1}{5}[10+3-0]=2.6$$

$$y^{(1)}=\frac{1}{2}\left[6-x^{(1)}\right]$$

$$=\frac{1}{2}[6-2.6]=1.7.$$

$$z^{(1)}=\frac{1}{5}\left[-1-x^{(1)}-y^{(1)}\right]$$

$$=\frac{1}{5}[-1-2.6-1.7]=-1.06.$$

Second iteration :-

Put $n=1$ in (1), we get

$$x^{(2)}=\frac{1}{5}\left[10+y^{(1)}-z^{(1)}\right]$$

$$=\frac{1}{5}[10+1.7+1.06]=2.552$$

$$y^{(2)} = \frac{1}{2}\left[6 - x^{(2)}\right]$$

$$= \frac{1}{2}\left[6 - 2.552\right] = 1.724$$

$$z^{(2)} = \frac{1}{5}\left[-1 - x^{(2)} - y^{(2)}\right]$$

$$= \frac{1}{5}\left[-1 - 2.552 - 1.724\right] = -1.0552$$

Third iteration :

Put $n = 2$ in (1), we get

$$x^{(3)} = \frac{1}{5}\left[10 + y^{(2)} - z^{(2)}\right]$$

$$= \frac{1}{5}\left[10 + 1.724 + 1.0552\right] = 2.55584$$

$$y^{(3)} = \frac{1}{2}\left[6 - x^{(3)}\right]$$

$$= \frac{1}{2}\left[6 - 2.55584\right] = 1.72208$$

$$z^{(3)} = \frac{1}{5}\left[-1 - x^{(3)} - y^{(3)}\right]$$

$$= \frac{1}{5}\left[-1 - 2.55584 - 1.72208\right] = -1.055584$$

Fourth iteration :-

Put $n = 3$ in (1), we get

$$x^{(4)} = \frac{1}{5}\left[10 + y^{(3)} - z^{(3)}\right]$$

$$= \frac{1}{5}\left[10 + 1.72208 + 1.055584\right] = 2.5555328$$

$$y^{(4)} = \frac{1}{2}\left[6 - x^{(4)}\right]$$

$$= \frac{1}{2}\left[6 - 2.5555328\right] = 1.7222336$$

$$z^{(4)} = \frac{1}{5}\left[-1 - x^{(4)} - y^{(4)}\right]$$

$$= \frac{1}{5}\left[-1 - 2.5555328 - 1.7222336\right] = -1.05555328$$

We observe that the solutions in the 3rd and 4th iterations are approxi-
-mately equal.

$\therefore$ The solution of the system is $x = 2.5555$, $y = 1.7222$, $z = -1.0556$

JACOBI'S ITERATION METHOD

1 Solve by Jacobi's iteration method, the equations $10x+y-z=11.19$, $x+10y+z=28.08$, $-x+y+10z=35.61$ correct to two decimal places

Ans:- $x=1.23$ $y=2.34$ $z=3.45$

2 Solve by Jacobi's iteration method, the equations $20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=25$ Ans:- $x=1, y=-1, z=1$.

3 Solve by Jacobi's method, the equations. $5x-y+z=10$, $2x+4y=12$, $x+y+5z=-1$. start with the solution $(2, 3, 0)$

Ans:- $x=2.556$, $y=1.722$, $z=-1.055$.

4 Solve $27x+6y-z=85$, $x+y+54z=110$, $6x+15y+2z=72$ by Jacobi's method. Ans:- $x=2.426$, $y=3.573$, $z=1.926$.

5 Solve $2x+y+6z=9$, $8x+3y+2z=13$, $x+5y+z=7$ by Jacobi's method.

Ans: $x=1$, $y=1$, $z=1$.

GAUSS SEIDEL ITERATION METHOD

1 Apply Gauss seidel iteration method it solve. $20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=25$ Ans:- $x=1$, $y=-1$, $z=1$.

2 Solve. $10x_1-2x_2-x_3-x_4=3$, $-2x_1+10x_2-x_3-x_4=15$, $-x_1-x_2+10x_3-2x_4=27$, $-x_1-x_2-2x_3+10x_4=-9$ by Gauss seidel iteration method.

Ans:- $x_1=1$ $x_2=2$ $x_3=3$ $x_4=0$.

3 Solve the following equations by Gauss seidel method.

$10x+y+z=12$, $2x+10y+z=13$, $2x+2y+10z=104$.

Ans:-

4 Solve. $83x+11y-4z=95$, $7x+52y+13z=104$, $3x+8y+29z=71$.

Ans:- $x=1.052$, $y=1.369$, $z=1.962$.

5 Solve by Gauss seidel iteration method. $28x+4y-z=32$, $x+3y+10z=24$, $2x+17y+4z=35$ Ans: $x=0.998$, $y=1.723$, $z=2.024$.

Condition for Convergence :-

We write $f(x)=0$ as $x=\phi(x)$ where $\phi(x)$ is also continuous in the interval in which the root lies.

We write the iteration method as $x_{k+1}=\phi(x_k)$, $k=0,1,2,\dots$ ——— ①

Let ξ be the exact root. That is $\xi=\phi(\xi)$ ——— ②

Define, error of approximation as $\varepsilon_k=\xi-x_k$, $k=0,1,2,\dots$ ——— ③

② − ① gives, $$\xi-x_{k+1}=\phi(\xi)-\phi(x_k).$$
$$=(\xi-x_k)\phi'(\alpha_k).$$
[∵ By using Lagrange's Mean value theorem]

$$\varepsilon_{k+1}=\varepsilon_k\phi'(\alpha_k),\quad x_k<\alpha_k<\xi.$$

Using this equation recursively, we get.
$$\varepsilon_k=\varepsilon_{k-1}\phi'(\alpha_{k-1}).$$
$$\varepsilon_{k+1}=\phi'(\alpha_k)\phi'(\alpha_{k-1})\varepsilon_{k-1}.$$
$$\varepsilon_{k-1}=\varepsilon_{k-2}\phi'(\alpha_{k-2})$$
$$\varepsilon_{k+1}=\phi'(\alpha_k)\phi'(\alpha_{k-1})\phi'(\alpha_{k-2})\varepsilon_{k-2}.$$
$$\dots\dots\dots\dots$$
$$\varepsilon_{k+1}=\phi'(\alpha_k)\phi'(\alpha_{k-1})\dots\phi'(\alpha_0)\cdot\varepsilon_0.$$

The initial error ε_0 is known and $|\varepsilon_0|$ is a finite quantity, we have
$$|\varepsilon_{k+1}|=|\phi'(\alpha_k)|\,|\phi'(\alpha_{k-1})|\dots|\phi'(\alpha_0)|\,|\varepsilon_0|.$$

Let $\phi'(\alpha_m)\le c$ $\quad m=0,1,2\dots k.$

Then $|\varepsilon_{k+1}|\le c^{k+1}|\varepsilon_0|.$

In the limit as $k\to\infty$, the right hand side $\to 0$, if and only if $c<1$.

Hence, the general iteration method converges to the root if
$|\phi'(x_k)|\le c<1$, $k=0,1,2\dots$

Rate of Convergence :—

The rate at which an iteration method converges if the initial approximation close to the exact root. Let x_k be the k^{th} iterate and ξ be the exact root. Then $\varepsilon_k = x_k - \xi$ is the error in the k^{th} iterate.

Order of the method :—

An iterative method is said to be order p or has the rate of convergence p, if p is the largest positive real number for which there exists a positive finite constant $c \neq 0$, such that

$$|\varepsilon_{k+1}| \leq c|\varepsilon_k|^p \quad \text{——} \quad (I)$$

The constant c is called the asymptotic error constant and depends on the derivatives of $f(x)$ at $x = \xi$

Equation (I) is also called the error equation.

Bisection Method :

Let the root of $f(x) = 0$ lie in the interval $I_0 = (a_0, b_0)$

We obtain a sequence of intervals $I_0 = (a_0, b_0)$, $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$.

Such that the length of the interval I_k is one half of the length of the interval I_{k-1}. For any k, we take the mid point of the interval I_k, that is $x_k = \frac{a_k + b_k}{2}$, as an approximation to the root

Since ξ lies in the interval (a_k, b_k) for all k,

We have $|\xi - x_{k+1}| \leq \frac{1}{2} |\xi - x_k|$

$$|\varepsilon_{k+1}| \leq \frac{1}{2} |\varepsilon_k|$$

Comparing above inequality with $|\varepsilon_{k+1}| \leq c |\varepsilon_k|^p$

We find that $p = 1$ and $c = \frac{1}{2}$

Hence we may conclude that the rate of convergence is 1.

that is the method has linear convergence.

Regula falsi method :—

Let $f(x)=0$ be the given equation.

Take two initial values x_0 and x_1, find $f(x_0), f(x_1)$

If $f(x_0)<0$, $f(x_1)>0$.

$\therefore$ The root lies between x_0 and x_1.

We take, one of the end points x_0 or x_1 of the initial interval (x_0, x_1) is always fixed and the other end point varies with k.

If the point x_0 is fixed, then the method is given by.

$$x_{k+1} = \frac{x_0 f_k - x_k f_0}{f_k - f_0}.$$

$$x_{k+1} = x_k - \frac{(x_k - x_0)}{(f_k - f_0)} . f_k .$$

Substituting $x_m = \xi + \varepsilon_m$, $m = 0, k, k+1$, in the Regula falsi method, we get

$$\xi + \varepsilon_{k+1} = \xi + \varepsilon_k - \left[\frac{\xi + \varepsilon_k - \xi - \varepsilon_0}{f(\xi+\varepsilon_k) - f(\xi+\varepsilon_0)}\right] f(\xi+\varepsilon_k)$$

$\left[\because f_k = f(x_k)\right.$

$\left. f_k = f(\xi+\varepsilon_k)\right]$

$$\varepsilon_{k+1} = \varepsilon_k - \left[\frac{\varepsilon_k - \varepsilon_0}{f(\xi+\varepsilon_k) - f(\xi+\varepsilon_0)}\right] f(\xi+\varepsilon_k) \quad \text{---} (1)$$

Since ξ is a simple root of $f(x)=0$, we have $f(\xi)=0$ and $f'(\xi)\neq 0$.

Expanding in Taylor series, we obtain.

[We know that the Taylor series.

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2} f''(a) + \dots \,]$$

$$f(\xi+\varepsilon_k) = f(\xi) + \frac{\varepsilon_k}{1!} f'(\xi) + \frac{\varepsilon_k^2}{2!} f''(\xi) + \dots$$

$$f(\xi+\varepsilon_k) = \frac{\varepsilon_k}{1!} f'(\xi) + \frac{\varepsilon_k^2}{2!} f''(\xi) + \dots \quad \text{---} (2) \qquad [\because f(\xi)=0]$$

$$f(\xi_r+\varepsilon_0) = f(\xi_r) + \frac{\varepsilon_0}{1!} f'(\xi_r) + \frac{\varepsilon_0^2}{2!} f''(\xi_r) + \cdots$$

$$f(\xi_r+\varepsilon_0) = \varepsilon_0 f'(\xi_r) + \frac{\varepsilon_0^2}{2!} f''(\xi_r) + \cdots \qquad [\because f(\xi_r)=0].$$

$$f(\xi_r+\varepsilon_k) - f(\xi_r+\varepsilon_0) = (\varepsilon_k-\varepsilon_0) f'(\xi_r) + \frac{1}{2} f''(\xi_r)(\varepsilon_k^2-\varepsilon_0^2) + \cdots$$

$$= (\varepsilon_k-\varepsilon_0)\left[f'(\xi_r) + \frac{1}{2}(\varepsilon_k+\varepsilon_0) f''(\xi_r) + \cdots\right]$$

$$= (\varepsilon_k-\varepsilon_0) f'(\xi_r)\left[1 + \frac{1}{2}(\varepsilon_k+\varepsilon_0)\frac{f''(\xi_r)}{f'(\xi_r)} + \cdots\right]$$

$$= (\varepsilon_k-\varepsilon_0) f'(\xi_r)\left[1 + \frac{1}{2}(\varepsilon_k+\varepsilon_0) A + \cdots\right]. \quad (3).$$

sub. (2),(3) in (1), we get

$$\varepsilon_{k+1} = \varepsilon_k - \frac{(\varepsilon_k-\varepsilon_0)\,[\varepsilon_k + A\varepsilon_k^2 + \cdots]}{(\varepsilon_k-\varepsilon_0) f'(\xi_r)[1+\frac{1}{2}(\varepsilon_k+\varepsilon_0)A + \cdots]} \qquad \left[\because A = \frac{f''(\xi_r)}{2f'(\xi_r)}\right]$$

$$\varepsilon_{k+1} = \varepsilon_k - [1 + A(\varepsilon_k+\varepsilon_0) + \cdots]^{-1}[\varepsilon_k + A\varepsilon_k^2 + \cdots]$$

$$\left[\text{WKT } (1+x)^{-1} = 1 - x + \frac{x^2}{2} - \cdots\right]$$

$$\varepsilon_{k+1} = \varepsilon_k - [1 - A(\varepsilon_k+\varepsilon_0) + \cdots][\varepsilon_k + A\varepsilon_k^2 + \cdots].$$

$$= \varepsilon_k - [\varepsilon_k - A\varepsilon_k\varepsilon_0 + \cdots]$$

$$\varepsilon_{k+1} = A\varepsilon_k\varepsilon_0.$$

$$|\varepsilon_{k+1}| = A_1|\varepsilon_k| \quad (4). \quad \text{where } A_1 = |A||\varepsilon_0|.$$

Note that ε_0 is fixed.

Compare (4) with $\varepsilon_{k+1} \le c|\varepsilon_k|^p$

Here $p=1$.

Hence, regula falsi method has linear rate of convergence.

Newton Raphson method :—

We know that the Newton Raphson method formula.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Substituting $x_k = \xi_r + \varepsilon_k$ in the Newton Raphson method.

$$\xi_r + \varepsilon_{k+1} = \xi_r + \varepsilon_k - \frac{f(\xi_r + \varepsilon_k)}{f'(\xi_r + \varepsilon_k)}$$

$$\varepsilon_{k+1} = \varepsilon_k - \frac{f(\xi_r + \varepsilon_k)}{f'(\xi_r + \varepsilon_k)}$$

[We know that the Taylor's series.

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots]$$

$$\varepsilon_{k+1} = \varepsilon_k - \frac{f(\xi_r) + \frac{\varepsilon_k}{1!} f'(\xi_r) + \frac{\varepsilon_k^2}{2!} f''(\xi_r) + \cdots}{f'(\xi_r) + \frac{\varepsilon_k}{1!} f''(\xi_r) + \frac{\varepsilon_k^2}{2!} f'''(\xi_r) + \cdots}$$

Since ξ_r is root of $f(x)=0$ then $f(\xi_r)=0$.

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\varepsilon_k f'(\xi_r) + \frac{\varepsilon_k^2}{2!} f''(\xi_r) + \cdots}{f'(\xi_r) + \frac{\varepsilon_k}{1!} f''(\xi_r) + \frac{\varepsilon_k^2}{2!} f'''(\xi_r) + \cdots}$$

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\varepsilon_k + \frac{\varepsilon_k^2}{2!} \frac{f''(\xi_r)}{f'(\xi_r)} + \cdots}{1 + \varepsilon_k \frac{f''(\xi_r)}{f'(\xi_r)} + \cdots}$$

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\varepsilon_k + A\varepsilon_k^2 + \cdots}{1 + 2\varepsilon_k A + \cdots}$$

$$\varepsilon_{k+1} = \varepsilon_k - [\varepsilon_k + A\varepsilon_k^2 + \cdots][1 + 2\varepsilon_k A + \cdots]^{-1}$$

We know that $(1+x)^{-1} = 1 - x + x^2 - \cdots$

$$\varepsilon_{k+1} = \varepsilon_k - [\varepsilon_k + A\varepsilon_k^2 + \cdots][1 - 2A\varepsilon_k + \cdots].$$

$$\varepsilon_{k+1} = \varepsilon_k - [\varepsilon_k - A\varepsilon_k^2 + \dots]$$

$$\varepsilon_{k+1} = A\varepsilon_k^2 + \dots$$

Neglecting the higher order terms, we get

$$\varepsilon_{k+1} = A\varepsilon_k^2 \quad (\text{or}) \quad |\varepsilon_{k+1}| \leq A_1 |\varepsilon_k|^2 \quad \text{where } A_1 = |A| \quad \text{——— (1).}$$

Compare (1) with $|\varepsilon_{k+1}| \leq c\,|\varepsilon_k|^p$

Here $p = 2$.

Hence the Newton Raphson method is of order 2 or has quadratic Convergence.

The error constant is given by $c = \left|\dfrac{f''(\xi)}{2f'(\xi)}\right|$.

Iteration method (Fixed point iteration method)

We write $f(x)=0$ as $x=\phi(x)$, where $\phi(x)$ is also continuous in the interval in which the root lies.

We write the iteration method as

$$x_{k+1} = \phi(x_k) \quad k=0,1,2,\ldots \qquad \text{——— (1)}$$

Let ξ be the exact root. That is

$$\xi = \phi(\xi) \qquad \text{——— (2)}.$$

Define, error of approximation as $\varepsilon_k = \xi - x_k$, $k=0,1,2,\ldots$

(2) − (1) gives,

$$\xi_k - x_{k+1} = \phi(\xi) - \phi(x_k)$$

$$= (\xi - x_k)\phi'(\alpha_k)$$

[Using Lagrange's mean value theorem]

$$\varepsilon_{k+1} = \phi'(\alpha_k)\,\varepsilon_k, \quad x_k < \alpha_k < \xi.$$

Using this equation recursively, we get

$$\varepsilon_k = \varepsilon_{k-1}\phi'(\alpha_{k-1}). \qquad \varepsilon_{k+1} = \phi'(\alpha_k)\,\phi'(\alpha_{k-1})\,\varepsilon_{k-1}$$

$$\varepsilon_{k-1} = \varepsilon_{k-2}\,\phi'(\alpha_{k-2})$$

$$\varepsilon_{k+1} = \phi'(\alpha_k)\,\phi'(\alpha_{k-1})\,\phi'(\alpha_{k-2})\,\varepsilon_{k-2}.$$

$$\cdots\cdots\cdots$$

$$\varepsilon_{k+1} = \phi'(\alpha_k)\,\phi'(\alpha_{k-1})\cdots\phi'(\alpha_0)\,\varepsilon_0.$$

The initial error ε_0 is known and $|\varepsilon_0|$ is a finite quantity, we have $|\varepsilon_{k+1}| = |\phi'(\alpha_k)|\,|\phi'(\alpha_{k-1})|\cdots|\phi'(\alpha_0)|\,|\varepsilon_0|$

Let $\phi'(\alpha_m) \le c$, $m=0,1,2,\ldots k$.

Then $|\varepsilon_{k+1}| \le c^{k+1}\,|\varepsilon_0|$

Now $|\varepsilon_{k+1}| \le c^{k+1}|\varepsilon_k| = A|\varepsilon_k|$ If $\varphi'(x_k) \neq 0$ i.e $\varphi'(\xi) \neq 0$.

The iterations converge If $|\varphi'(\xi)| \le c < 1$.

In this case, the method has linear rate of convergence.

Now, let $\varphi'(\xi) = 0$ and $\varphi''(\xi) \neq 0$.

Then, since $\xi = \varphi(\xi)$, we get

$$x_{k+1} - \xi = \varphi(x_k) - \xi.$$

$$= \varphi[\xi + (x_k - \xi)] - \xi.$$

$$= \left[\varphi(\xi) + (x_k - \xi)\varphi'(\xi) + \frac{(x_k - \xi)^2}{2!}\varphi''(\xi) + \cdots\right] - \xi$$

(∵ By Taylor's theorem)

$$= \cancel{\xi} + (x_k - \xi)\varphi'(\xi) + \frac{(x_k - \xi)^2}{2!}\varphi''(\xi) + \cdots - \cancel{\xi}$$

$$= \frac{1}{2}(x_k - \xi)^2 \varphi''(\xi) + \cdots$$

Neglecting the higher order terms, we get

$$|\varepsilon_{k+1}| \le \frac{1}{2}|\varphi''(\xi)|\,|\varepsilon_k|^2.$$

Hence the method is of order 2 or has quadratic convergence.

Now let $\xi = \varphi(\xi)$, $\varphi'(\xi) = 0 = \cdots = \varphi^{(p-1)}(\xi)$, $\varphi^{(p)}(\xi) \neq 0$.

Then, from the above equation, we get

$$|\varepsilon_{k+1}| \le \frac{1}{p!}|\varphi^{(p)}(\xi)|\,|\varepsilon_k|^p$$

Hence the method is of order p.

MODULE-IV

INTERPOLATION

Interpolation :-

Interpolation is a method of constructing new data points from a discrete set of known data points.

i.e. Interpolation is the process of finding out the unknown value which lies in the given set of tabulated values.

Extrapolation :-

Extrapolation is the process of finding out the unknown value which lies outside the given set of tabulated values.

Introduction :-

Let $y = f(x)$ $x_0 \le x \le x_n$ be defined. We can find the value of y for all values of x because $y = f(x)$ is defined explicitely.

But without having the explicit definition of $y = f(x)$, it is diffi-cult to find y.

If we have a set of tabular values

x	x_0	x_1	x_2	....	x_n
y	y_0	y_1	y_2	....	y_n

Which satisfy $y = f(x)$ then we can find the value of y for the corresponding value of x, by using interpolation.

Finite differences :—

Let $y = f(x)$ be a function in $x_0 \le x \le x_n$, x_i are equally spaced (i.e the difference between x_i and x_{i+1} is same $\forall\, i = 0, 1, 2 \ldots n-1$)

Then we can recover the values of y for some intermediate values of x in the range $x_0 \le x \le x_n$ by using the differences of $f(x)$

The first finite difference of y is

$$\Delta y = \Delta f(x)$$

$$\Delta y = f(x + \Delta x) - f(x) \text{ where } \Delta x \text{ is the increment in } x.$$

$$\begin{aligned}
\Delta^2 y &= \Delta(\Delta y) \\
&= \Delta[f(x + \Delta x) - f(x)] \\
&= \Delta f(x + \Delta x) - \Delta f(x) \\
&= f(x + 2\Delta x) - f(x + \Delta x) - f(x + \Delta x) + f(x)
\end{aligned}$$

$$\Delta^2 y = f(x + 2\Delta x) - 2f(x + \Delta x) + f(x).$$

In general $\Delta^n y = \Delta(\Delta^{n-1} y)$ for $n = 2, 3, 4, \ldots$

Forward Differences :—

Let $y = f(x)$ be the continuous function. Let $y_0, y_1, y_2 \ldots y_n$ be the corrosponding values of y at $x = x_0, x_1, x_2 \ldots x_n$ respectively.

Then the differences $y_1 - y_0,\ y_2 - y_1,\ y_3 - y_2 \ldots y_n - y_{n-1}$

i.e $y_i - y_{i-1}$ $\forall\, i = 1, 2, 3 \ldots n$ are called the first forward differe-nces.

They are denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots \Delta y_{n-1}$

$$\Delta y_0 = y_1 - y_0 \quad \Delta y_1 = y_2 - y_1 \quad \Delta y_2 = y_3 - y_2 \quad \Delta y_{n-1} = y_n - y_{n-1}$$

Where Δ is called the forward difference operator.

The difference of the first forward differences are called second forward differences and they are denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2 \ldots$

Backward Differences :-

Let $y = f(x)$ be the continuous function. Let $y_0, y_1, y_2 \ldots y_n$ be the corresponding values of y at $x = x_0, x_1, x_2 \ldots x_n$ respectively. The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2 \ldots y_n - y_{n-1}$ are called the first backward differences. They are denoted by $\nabla y_1, \nabla y_2, \nabla y_3 \ldots \nabla y_n$ respectively where ∇ is called the backward difference operator.

$$\nabla y_1 = y_1 - y_0 \quad \nabla y_2 = y_2 - y_1 \quad \nabla y_3 = y_3 - y_2 \quad \ldots \quad \nabla y_n = y_n - y_{n-1}$$

Generally $\nabla y_n = y_n - y_{n-1}$.

The differences of the first backward differences are called second backward differences and they are denoted by $\nabla^2 y_2, \nabla^2 y_3, \ldots \nabla^2 y_n$.

Similarly we can define higher backward differences.

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0$$

The Backward Difference Table :-

x	y	∇	∇^2	∇^3
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$
x_3	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	
		$\nabla y_4 = y_4 - y_3$		
x_4	y_4			

Similarly we can define higher order forward differences

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = y_3 - y_2 - (y_2 - y_1) = y_3 - 2y_2 + y_1$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

The Forward Difference Table :-

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_1$
		$\Delta y_3 = y_4 - y_3$		$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$	
x_4	y_4		$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$		
		$\Delta y_4 = y_5 - y_4$			
x_5	y_5				

Note :- (a) Δ can also be defined as $\Delta f(x) = f(x+h) - f(x)$

(b) $\Delta f(x) = \Delta K = 0$ [$\because f(x) = K$ is a constant function
Then difference of a constant function is zero]

(c) $\Delta(u_K + v_K) = \Delta(u_K) + \Delta(v_K)$

(d) $\Delta(u_K v_K) = u_K \Delta(v_K) + v_K \Delta(u_K)$

Central Differences :—

The central difference operator δ defined as

$\delta y_{\frac{1}{2}} = y_1 - y_0 \qquad \delta y_{\frac{3}{2}} = y_2 - y_1 \qquad \delta y_{\frac{5}{2}} = y_3 - y_2 \qquad \ldots \qquad \delta y_{\frac{2n-1}{2}} = y_n - y_{n-1}$.

Similarly higher order central differences can be defined as

$$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} = y_2 - 2y_1 + y_0$$

$$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}} = y_3 - 2y_2 + y_1$$

x	y	δ	δ^2	δ^3
x_0	y_0			
		$\delta y_{\frac{1}{2}} = y_1 - y_0$		
x_1	y_1		$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$	
		$\delta y_{\frac{3}{2}} = y_2 - y_1$		$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$
x_2	y_2		$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$	
		$\delta y_{\frac{5}{2}} = y_3 - y_2$		$\delta^3 y_{\frac{5}{2}} = \delta^2 y_3 - \delta^2 y_2$
x_3	y_3		$\delta^2 y_3 = \delta y_{\frac{7}{2}} - \delta y_{\frac{5}{2}}$	
		$\delta y_{\frac{7}{2}} = y_4 - y_3$		$\delta^3 y_{\frac{7}{2}} = \delta^2 y_4 - \delta^2 y_3$
x_4	y_4		$\delta^2 y_4 = \delta y_{\frac{9}{2}} - \delta y_{\frac{7}{2}}$	
		$\delta y_{\frac{9}{2}} = y_5 - y_4$		
x_5	y_5			

• It is clear from the three tables that in a definite numerical case, the same numbers occur in the same positions whether we use forward, backward or central differences.

Then we obtain $\Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$

(1) Given that $u_0 = 1$ $u_1 = 11$ $u_2 = 21$ $u_3 = 28$ $u_4 = 29$ find $\Delta^4 u_0$

x	u	Δ	Δ^2	Δ^3	Δ^4
0	1				
		10			
1	11		0		
				-3	
		10			0
			-3		
2	21				
				-3	
		7			
3	28		-6		
		1			
4	29				

(2) Given that $u_0 = 3$ $u_1 = 12$ $u_2 = 81$ $u_3 = 200$ $u_4 = 100$ $u_5 = 8$.

find $\Delta^5 u_0$.

(3) If $f(x) = x^3 + 5x - 7$ form a table of forward difference taking $x = -1, 0, 1, 2, 3, 4, 5$ show that the third differences are constant.

(4) Construct a forward difference table from the following data.

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

Write the values of $\Delta f(10)$, $\Delta^2 f(10)$, $\Delta^3 f(15)$, and $\Delta^4 f(15)$

Newton's Forward Interpolation Formula :-

Let $y = f(x)$ be a function. At $x = x_0, x_1, x_2, \ldots x_n$ let the corrospondi values of y be $y_0\ y_1\ y_2 \ldots y_n$. Here x values are equally spaced with common difference h.

Then Newton's forward interpolation formula is given by.

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots + \frac{P(P-1)(P-2)\cdots(P-(n-1))}{n!}\Delta^n y_0$$

$$\text{Where} \quad P = \frac{x - x_0}{h}$$

Note :- It is used to interpolate the values of y nearer to the begining of a set of tabular values.

(1) Find the cubic polynomial which takes the values.

$y(0) = 1 \quad y(1) = 0 \quad y(2) = 1 \quad y(3) = 10.$

Sol:- The forward difference table is

x	y	Δ	Δ^2	Δ^3
0	1			
		$-1 = \Delta y_0$		
1	0		$2 = \Delta^2 y_0$	
		1		$6 = \Delta^3 y_0$
2	1		8	
		9		
3	10			

Here $x_0 = 0 \quad y_0 = 1 \quad h = 1$

$$P = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

$\Delta y_0 = -1 \quad \Delta^2 y_0 = 2 \quad \Delta^3 y_0 = 6.$

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 .$$

$$y = 1 + x(-1) + \frac{x(x-1)}{2!}2 + \frac{x(x-1)(x-2)}{3!}6$$

$$y = 1 - x + x^2 - x + x^3 - 3x^2 + 2x$$

$$y = x^3 - 2x^2 + 1$$

(2) Find the Newton's forward difference interpolating polynomial for the data.

x	0	1	2	3
y	1	3	7	13

Ans:- $y = x^2 + x + 1$

(3) Construct the difference table for data and then express y as a function of x.

x	0	1	2	3	4
y	3	6	11	18	27

Ans:- $y = x^2 + 2x + 3$

(4) Using Newton's forward interpolation formula, find a polynomial of degree 2 which takes the values.

x	4	6	8	10
y	1	3	8	16

Ans: $\frac{3x^2}{8} - \frac{11x}{4} + 6$.

(5) Find the Newton's forward difference interpolating polynomial for the data

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

(1) The values of sinx are given below for different values of x. (7) (a)

Find sin32.

x	30°	35°	40°	45°	50°
y = sinx	0.5	0.5736	0.6428	0.7071	0.7660

Sol:- The value of sin32 is nearer to the begining of the table.

∴ We use Newton's forward difference interpolation formula.

The forward Difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
$30° = x_0$	$0.5 = y_0$				
		Δy_0 0.0736			
35°	0.5736		$\Delta^2 y_0$ −0.0044		
		0.0692		$\Delta^3 y_0$ −0.0005	
40	0.6428		−0.0049		$\Delta^4 y_0$ 0
		0.0643		−0.0005	
45°	0.7071		−0.0054		
		0.0589			
50°	0.7660				

The Newton's forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$$

where $P = \frac{x - x_0}{h}$.

$x_0 = 30,\ y_0 = 0.5 \quad \Delta y_0 = 0.0736 \quad \Delta^2 y_0 = -0.0044 \quad \Delta^3 y_0 = -0.0005 \quad \Delta^4 y_0 = 0$

$h = 5 \quad x = 32 \qquad P = \frac{32-30}{5} = \frac{2}{5} = 0.4$

$$y = 0.5 + (0.4)(0.0736) + \frac{(0.4)((0.4)-1)}{2!}(-0.0044) + \frac{0.4(0.4-1)(0.4-2)}{3!}(-0.0005) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(0)$$

$$y = 0.5 + 0.02944 + 0.000528 - 0.000032 = 0.529936$$

15) Construct the forward difference table for data and then express y as a function of x.

x	0	1	2	3	4
y	1	3	9	31	81

Sol: The Forward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1	$\Delta y_0 = 2$	$4 = \Delta^2 y_0$	$12 = \Delta^3 y_0$	0
1	3	6	16	12	
2	9	22	28		
3	31	50			
4	81				

$x_0 = 0 \quad y_0 = 1 \quad \Delta y_0 = 2 \quad \Delta^2 y_0 = 4 \quad \Delta^3 y_0 = 12.$

$h = 1 \quad P = \dfrac{x - x_0}{h} = \dfrac{x - 0}{1} = x$

The Newton's forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$y = 1 + x.2 + \frac{x(x-1)}{\cancel{2!}}\cancel{4}^{2} + \frac{x(x-1)(x-2)}{\cancel{3!}}\cancel{12}^{2}$$

$$y = 1 + 2\cancel{x} + 2x^2 - 2\cancel{x} + 2x(x^2 - 3x + 2)$$

$$y = 1 + 2x^2 + 2x^3 - 6x^2 + 4x$$

$$y = 2x^3 - 4x^2 + 4x + 1$$

(2) Given that $\sqrt{12500} = 111.8034$ $\sqrt{12510} = 111.8481$ $\sqrt{12520} = 111.8928$

$\sqrt{12530} = 111.9375$ Find $\sqrt{12516}$

Sol:- The value of $\sqrt{12516}$ is nearer to the begining of the table

$\therefore$ We use the Newton's forward interpolation formula.

The forward difference table is

x	y	Δ	Δ^2	Δ^3
12500	111.8034			
		0.0447		
			0	
12510	111.8481			0
		0.0447		
			0	
12520	111.8928			
		0.0447		
12530	111.9375			

$x_0 = 12500$ $y_0 = 111.8034$ $\Delta y_0 = 0.0447$ $\Delta^2 y_0 = 0$ $\Delta^3 y_0 = 0$.

$h = 10$.

The Newton's forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0.$$

where $P = \frac{x - x_0}{h}$

$x = 12516$, $P = \frac{12516 - 12500}{10} = \frac{16}{10} = 1.6$

$$y = 111.8034 + 1.6(0.0447) + \frac{1.6(1.6-1)}{2!}(0) + \frac{1.6(1.6-1)(1.6-2)}{3!}(0)$$

$y = 111.8034 + 0.07152$

$y = 111.87492$

$\therefore y = \sqrt{12516} = 111.87492$

(3) From the following table, find the no. of students who obtained less than 45 marks.

Marks	No. of students
30 - 40	31
40 - 50	42
50 - 60	51
60 - 70	35
70 - 80	31

sol:- 45 The forward difference table is

Marks less than x	No. of Students y	Δ	Δ^2	Δ^3	Δ^4
40	31				
		42			
			9		
50	73			-25	
		51			
			-16		37
60	124				
				12	
		35			
			-4		
70	159				
		31			
80	190				

45 is nearer to the begining of the table.

$\therefore$ we use the Newton's forward interpolation formula.

Newton's forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0.$$

where $P = \frac{x - x_0}{h}$

(8)

$y_0 = 31 \quad \Delta y_0 = 42 \quad \Delta^2 y_0 = 9 \quad \Delta^3 y_0 = -25 \quad \Delta^4 y_0 = 37.$

$x_0 = 40 \quad x = 45 \quad h = 10. \quad p = \frac{45-40}{10} = \frac{5}{10} = 0.5$

$$y = 31 + (0.5)(42) + \frac{(0.5)(0.5-1)}{2!}(9) + \frac{0.5(0.5-1)(0.5-2)}{3!}(-25)$$

$$\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$$

$y = 31 + 21 - 1.125 - 1.5625 - 1.4453125$

$y = 47.8672$

$y = 48$ (approximately.

From the data given below, find the numbes of students whose weight is between 60 and 70.

Weight in Kgs	0 – 40	40 – 60	60 – 80	80 – 100	100 – 120
No. of students	250	120	100	70	50

Sol: The cumulative values and the differenc table is given by

Weight x	No. of students y	Δ	Δ^2	Δ^3	Δ^4
Below 40 x_0	250 y_0				
		120 Δy_0			
60	370		-20 $\Delta^2 y_0$		
				-10 $\Delta^3 y_0$	
		100			20 $\Delta^4 y_0$
			-30		
80	470				
				10	
		70			
			-20		
100	540				
		50			
120	590				

The Newtons forward interpolation formula is given by

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

Where $p = \dfrac{x - x_0}{h}$.

Here $x_0 = 40$, $x = 70$ $h = 20$ $\Delta y_0 = 120$ $\Delta^2 y_0 = -20$

$\Delta^3 y_0 = -10$ $\Delta^4 y_0 = 20$

$y_0 = 250$.

$$p = \frac{70-40}{20} = 1.5$$

$$y = 250 + (1.5)(120) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{3!}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{4!}(20).$$

$$y = 250 + 180 - 7.5 + 0.625 + 0.46875 = 423.59 = 424$$

No. of students whose weight is between 60 and 70.

$$y(70) - y(66) = 424 - 370 = 54.$$

NEWTONS FORWARD INTERPOLATION FORMULA.

(1) Using Newton's forward interpolation formula, find a second degree polynomial passes through the points (1, −1) (2, −1) (3, 1) and (4, 5).

Ans:- $x^2 - 3x + 1$

(2) Find a cubic polynomial which takes the following values.

x	0	1.	2	3
y	1	2	1	10

Ans:- $2x^3 - 7x^2 + 6x + 1$

(3) Find a Newton's forward interpolation polynomial for the data.

x	4	6	8	10
y	1	3	8	16

Ans:- $\frac{1}{8}(3x^2 - 22x + 48)$.

(4) Find an interpolating polynomial for the function $f(x)$.

x	0	2	4	6	8	10
y	0	4	56	204	496	980

Ans:- $x^3 - 2x$.

(5) Using the Newton's forward differences formula, find the interpolating polynomial for the function $y = f(x)$ given by $f(0) = 1$, $f(1) = 2$, $f(2) = 1$ $f(3) = 10$. Hence evaluate $f(0.75)$ and $f(-0.5)$.

Ans:- $2x^3 - 7x^2 + 6x + 1$, 2.40625, −4.

(6) Given $f(0) = 1$, $f(1) = 0$ $f(2) = 1$ $f(3) = 10$, find an interpolating polynomial for $f(x)$ using the Newtons forward interpolation formula. Hence evaluate $f(0.4)$. Ans:- $x^3 - 2x^2 + 1$, 0.744

(7) From the data given in the following table, find the number of students who obtained (i) less than 45 marks (ii) between 40 and 45 marks.

Marks	30−40	40−50	50−60	60−70	70−80
No. of students	31	42	51	35	31

Ans:- 48, 17.

(8) Given $\sin 10^\circ = 0.17365$, $\sin 11^\circ = 0.19081$, $\sin 12^\circ = 0.20791$, $\sin 13^\circ = 0.22493$ (3)

determine $\sin(10^\circ 21')$ using the Newton's forward formula.

Ans:- 0.179377

(9) From the following table of values of $y = f(x)$, find the values of y for $x = 3.25$, $x = 3.5$ and $x = 3.75$.

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Ans:- 5.493, 6.319, 7.285

(10) A function $y = f(x)$ is given by the following table.

x	1.0	1.2	1.4	1.6	1.8	2.0
$y = f(x)$	0.00	0.128	0.544	1.296	2.432	4.00

Find an approximate value of $f(1.1)$.

(11) The following table gives a set of values of $f(x) = \frac{\sin x}{x^2}$. Using this Table, find an approximate value of $\sin(0.15)$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	9.9833	4.9696	3.2836	2.4339	1.9177

Ans:- 0.1495

(12) Given $\log_{10} 654 = 2.8156$, $\log_{10} 656 = 2.8169$, $\log_{10} 658 = 2.8182$, $\log_{10} 660 = 2.8195$

$\log_{10} 662 = 2.821$, find $\log_{10} 655$. Ans:- 2.8162

(13) Estimate the value of $\tan(0.12)$

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Ans:- 0.1205

(14) Given $f(1) = 3.49$, $f(1.4) = 4.82$, $f(1.8) = 5.96$ $f(2.2) = 6.5$ Using the Newtons forward interpolation formula, find $f(1.6)$

Ans:- 5.4396

Note :- Let the given tabular values of unknown function $y = f(x)$ is

x	x_0	x_1	x_2	x_3	x_i	x_{i+1}	x_{i+2}	x_{n-1}
$y = f(x)$	y_0	y_1	y_2	y_3	y_i	y_{i+1}	y_{i+2}	y_{n-1}

(i) For finding one missing term y_i at $x = x_i$ in given table, we equate $(n-1)^{th}$ forward or backward difference to zero, then we solve the resultant equation. Here $n-1$ = No. of known y values.

(ii) For finding two missing terms y_i and y_{i+j} at $x = x_i$ and $x = x_{i+j}$ in given table we equate $(n-2)^{nd}$ forward or backward differences to zero then we solve the resultant equations.

Here $n-2$ = No. of known y values.

1) Estimate the missing term in the following table

x	0	1	2	3	4
y	1	3	9	—	81

Sol:- The forward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		2			
1	3		4		
		6		$P-19$	
2	9		$P-15$		$124-4P$
		$P-9$		$105-3P$	
3	P		$90-2P$		
		$81-P$			
4	81				

Let us consider $\Delta^4 y_0 = 0$

$$124 - 4P = 0$$

$$4P = 124$$

$$P = 31$$

Find the missing values in the following data.

x	45	50	55	60	65
y	3	—	2	—	-2.4

Sol. Let the missing value be a, b. Then the difference table is.

x	y	Δ	Δ^2	Δ^3
45	3			
		$a-3$		
50	a		$5-2a$	
		$2-a$		$3a+b-9$
55	2		$a+b-4$	
		$b-2$		$3.6-a-3b$
60	b		$-0.4-2b$	
		$-2.4-b$		
65	-2.4			

As only three entries $y_0\ y_2\ y_4$ are given, y can be represented by a 2nd degree polynomial having third differences as zero.

$$\therefore\ \Delta^3 y_0 = 0 \quad \Delta^3 y_1 = 0$$

$$3a+b=9, \quad a+3b=3.6$$

Solving these eqn's, we get $a = 2.925$ $b = 0.225$

II Method :-

As only three entries $y_0 = 3$ $y_2 = 2$ $y_4 = -2.4$ are given, y can be repre-sented by a second degree polynomial having third differences as zero.

$$\Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

$$(E-1)^3 y_0 = 0 \quad \text{and} \quad (E-1)^3 y_1 = 0.$$

$$E^3 y_0 - 3E^2 y_0 + 3E y_0 - y_0 = 0, \quad E^3 y_1 - 3E^2 y_1 + 3E y_1 - y_1 = 0.$$

$$y_3 - 3y_2 + 3y_1 - y_0 = 0, \quad y_4 - 3y_3 + 3y_2 - y_1 = 0.$$

$$y_3 + 3y_1 = 9 \qquad 3y_3 + y_1 = 3.6$$

Solving these equations, we get

$$y_1 = 2.925 \quad y_3 = 0.225$$

Find the missing term in the table

x	2	3	4	5	6
y	45	49.2	54.1	—	67.4

Sol: Let the missing term be p. Then the forward difference table is.

x	y	Δ	Δ^2	Δ^2	Δ^4
2	45				
		4.2			
			0.7		
3	49.2			$p-59.7$	
		4.9			$240.2-4p$
			$p-59$		
4	54.1			$180.5-3p$	
		$p-54.1$			
			$121.5-2p$		
5	p				
		$67.4-p$			
6	67.4				

We know that $\Delta^4 y_0 = 0$ i.e. $240.2 - 4p = 0$

$$p = 60.05.$$

II Method :—

As only four entries $y_0\ y_1\ y_2\ y_3$ are given therefore $y = f(x)$ can be represented by a third degree polynomial.

$$\therefore \Delta^3 y = \text{Constant}.$$

$$\text{or} \quad \Delta^4 y_0 = 0$$

$$\text{i.e.} \quad (E-1)^4 y_0 = 0$$

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0.$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0.$$

Let the missing entry y_3 be p so that

$$67.4 - 4p + 6(54.1) - 4(49.2) + 45 = 0$$

$$-4p = -240.2$$

$$p = 60.05$$

INTERPOLATION.

(1) Find the missing value in the following table

(a)

x	0	1	2	3	4
y	1	2	4	–	15

Ans:- 8

(b)

x	1	2	3	4	5
y	2	–	40	83	150

Ans:- 15

(c)

x	1	2	3	4	5
y	15.75	17.9	–	22.75	43.2

Ans:- 1

(2) Find the missing terms in the following table.

(a)

x	1	2	3	4	5	6	7
y	103.4	97.6	122.9	–	179.0	–	195.8

Ans. 154.8575
190.825

(b)

x	6	7	8	9	10	11	12
y	0.77815	–	0.90309	0.95424	1	–	1.07918

Ans. 0.84494
1.04146

(c)

x	45	50	55	60	65
y	3	–	2	–	−2.4

Ans:- 2.325
2.025

(3) If $y_0 = 4$, $y_1 = 8$ $y_2 = 21$ $y_3 = 75$ $y_4 = 32$ $y_5 = 16$ and $y_6 = 10$ find $\Delta^6 y_0$ without forming the Difference table.

Newton's Backward Interpolation formula :—

Let $y = f(x)$ be a function. Let $y_0, y_1, y_2, y_3, \ldots y_n$ be the values of y at $x = x_0, x_1, x_2, \ldots x_n$. These x values are equally spaced with common difference h.

Then the Newton's Backward Interpolation formula is given by

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p(p+1)\ldots(p+(n-1))}{n!}\nabla^n y_n$$

where $p = \frac{x - x_n}{h}$.

<u>Note</u>:- The Newton's Backward interpolation formula is used for interpolating a values of y nearer to the end of the table of values

(1) Use Newton's Backward interpolation formula to find the polynomial satisfied by $(3, 6)$ $(4, 24)$ $(5, 60)$ and $(6, 120)$

Sol:- The Backward difference table is :

x	y	∇	∇^2	∇^3
3	$6 = y_0$			
		$18 = \nabla y_1$		
4	$24 = y_1$		$18 = \nabla^2 y_2$	
		$36 = \nabla y_2$		$6 = \nabla^3 y_3$
5	$60 = y_2$		$24 = \nabla^2 y_3$	
		$60 = \nabla y_3$		
6	$120 = y_3$			

The Newton's Backward Difference interpolation formula is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

where $p = \frac{x - x_n}{h}$

$x_n = 6 \quad y_n = 120 \quad \nabla y_n = 60 \quad \nabla^2 y_n = 24 \quad \nabla^3 y_n = 6$

$h = 1 \quad p = \frac{x - x_n}{h} = \frac{x-6}{1} = x-6$

$y = 120 + (x-6)\,60 + \frac{(x-6)(x-5)}{2} 24 + \frac{(x-6)(x-5)(x-4)}{6} \cdot 6$

$y = x^3 - 3x^2 + 2x$

(2) Find y at 9 from the following table.

x	2	5	8	11
y	94.8	87.9	81.3	75.1

Sol:- The value of y at $x=9$ is nearer to the ending of the table.

$\therefore$ We use Newton's Backward interpolation formula.

The Backward Difference Table is.

x	y	∇	∇^2	∇^3
2	$94.8 = y_0$			
		$-6.9 = \nabla y_1$		
5	$87.9 = y_1$		$0.3 = \nabla^2 y_2$	
		$-6.6 = \nabla y_2$		$0.1 = \nabla^3 y_3$
8	$81.3 = y_2$		$0.4 = \nabla^2 y_3$	
		$-6.2 = \nabla y_3$		
11	$75.1 = y_3$			

The Newton's Backward interpolation formula is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

where $p = \frac{x - x_n}{h}$

$x_n = 11,\quad y_n = 75.1 \quad \nabla y_n = -6.2 \quad \nabla^2 y_n = 0.4 \quad \nabla^3 y_n = 0.1$

$$p = \frac{x - x_n}{h} = \frac{9-11}{3} = -\frac{2}{3}.$$

$$y = 75.1 - \frac{2}{3}(-6.2) + \frac{-\frac{2}{3}\left(-\frac{2}{3}+1\right)}{2!}(0.4) + \frac{-\frac{2}{3}\left(-\frac{2}{3}+1\right)\left(-\frac{2}{3}+2\right)}{3!}(0.1)$$

$$y = 75.1 + 4.133 - 0.044 - 0.005$$

$$y = 79.184.$$

13) Calculate the value $f(7.5)$ for the table.

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Sol:- The value of $f(x)$ at $x = 7.5$ is nearer to the ending of the table.

$\therefore$ We use Newton's Backward interpolation formula.

The Backward Difference Table is.

x	y	∇	∇^2	∇^3	∇^4	∇^5
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		0
5	125		30		0	
		91		6		0.
6	216		36		0	
		127		6		
7	343		42			
		169				
8	512					

The Newton's Backward interpolation formula is

$$y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \frac{P(P+1)(P+2)(P+3)}{4!}\nabla^4 y_n$$

where $P = \frac{x - x_n}{h}$

$x_n = 8 \quad y_n = 512 \quad h = 1, \quad \nabla y_n = 169 \quad \nabla^2 y_n = 42 \quad \nabla^3 y_n = 6$

$$P = \frac{7.5-8}{1} = \frac{-0.5}{1} = -0.5$$

$$y = 512 + (-0.5)169 + \frac{(-0.5)(-0.5+1)}{2!}\cancel{27}\,42 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(6)$$

$$y = 512 - 84.5 - 5.25 - 0.375$$

$$y = 421.875$$

(4) In the table, the values of y are consecutive terms of a series of which the number 21.6 is the 6th term. Find the first and tenth terms of the results.

x	3	4	5	6	7	8	9
y	2.7	6.4	12.5	21.6	34.3	51.2	72.9

Sol:- The Difference Table is

x	y	Δ	Δ^2	Δ^3	Δ^4
3	2.7				
		3.7			
4	6.4		2.4		
		6.1		0.6	
5	12.5		3.0		0
		9.1		0.6	
6	21.6		3.6		0
		12.7		0.6	
7	34.3		4.2		0.
		16.9		0.6	
8	51.2		4.8		
		21.7			
9	72.9				

To find 10^{th} term :—

$x_n = 9 \quad y_n = 72.9 \quad \nabla y_n = 21.7, \quad \nabla^2 y_n = 4.8 \quad \nabla^3 y_n = 0.6$

$h = 1$

The Newton's Backward interpolation formula is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

where $p = \frac{x - x_n}{h}$

$p = \frac{10-9}{1} = 1.$

$$y = 72.9 + (1)(21.7) + \frac{(1)(1+1)}{2}(4.8) + \frac{1(1+1)(1+2)}{3!}(0.6)$$

$$y = 72.9 + 21.7 + 4.8 + 0.6$$

$$y = 100.$$

To find 1^{st} term :—

$x_0 = 3 \quad y_0 = 2.7, \quad \Delta y_0 = 3.7 \quad \Delta^2 y_0 = 2.4 \quad \Delta^3 y_0 = 0.6.$

$h = 1$

The Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

where $p = \frac{x - x_0}{h}$.

$$p = \frac{1-3}{1} = -2$$

$$y = 2.7 + (-2)\,3.7 + \frac{(-2)(-2-1)}{2!}(2.4) + \frac{(-2)(-2-1)(-2-2)}{3!}(0.6).$$

$$y = 2.7 - 7.4 + 7.2 - 2.4$$

$$y = 0.1$$

14) The area A of a circle of diameter d is given below.

d	80	85	90	95	100.
A	5026	5674	6362	7088	7854

Find approximately the area of circles of diameters 82 and 91.

Ans:- 5280.1056, 6504.1536.

15) Find following data is taken from steam table. Find the pressure at temperature. $t = 142^\circ$, $t = 175^\circ$

Temp °C	140	150	160	170	180
Pressure kg/cm^2	3.685	4.854	6.302	8.076	10.225

Ans:- 3.898, 4.100.

NEWTONS BACKWARD INTERPOLATION FORMULA.

(1) Using Newton's backward interpolation formula, find the interpolating polynomial for the function given by the following table.

x	10	11	12	13
$y=f(x)$	21	23	27	33

Ans:- $x^2-19x+111$
29.75

Hence find $f(12.5)$.

(2) Using Newton's backward interpolation formula, find the interpolating polynomial for the function given $y=f(x)$.

$f(0)=1$ $f(1)=2$ $f(2)=1$ $f(3)=10$. Hence find $f(2.5)$

Ans:- $2x^3-7x^2+6x+1$, 3.5

(3) Using Newton's backward interpolation formula; find the interpolating polynomials for the functions given by the following tables.

(a)

x	0	1	2	3
$f(x)$	1	3	7	13

Ans:- x^2+x+1

(b)

x	0	1	2	3	4
$f(x)$	-5	-10	-9	4	35

Ans: x^3-6x-5

(c)

x	-4	-2	0	2	4
$f(x)$	-25	1	3	29	127

Ans:- x^3+3x^2+3x+3.

(4) From the following table, estimate the number of students who obtained marks between 76 and 80.

Marks	36-45	46-55	56-65	66-75	76-85
No. of Students	18	40	64	50	28

Ans:- 16.

(5) Given $f(40)=184$, $f(50)=204$ $f(60)=226$ $f(70)=250$ $f(80)=276$ $f(90)=304$ find $f(85)$. Ans:- 289.75.

(6) The population of a certain town is given by the following ... 5

Year	1961	1971	1981	1991	2001
Population (in thousands)	19.96	39.65	58.81	77.18	94.58

Using Newton's forward and backward interpolation formulas, find the increase in the population from the year 1965 to 1995

Ans:- 27.8796, 84.264, 56.38.

(7) The deflection d measured at various distances x from one end of a cantilever is given by the following table.

x	0.0	0.2	0.4	0.6	0.8	1.0
d	0.00	0.0347	0.1173	0.2160	0.2987	0.3333

Find the value of d for $x = 0.1$ and $x = 0.95$

Ans:- , 0.3306.

(8). Find y when $x = 19.5, 23.4$ and 24.5

x	19	20	21	22	23	24	25
y	91.00	100.25	110.00	120.25	131.00	142.55	154.00

Ans:- , 135.44 ,

(9) Estimate the values of $f(42)$, $f(44)$ and $f(21)$.

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

Ans:- 219, ,

(10) Find $f(42)$ and $f(84)$ using the following table.

x	40	50	60	70	80	90
$f(x)$	184	204	226	250	276	304

Ans:- 187.84, 287.

Central Difference Interpolation Formula :—

Newton's forward interpolation formula is useful to find the value of $y = f(x)$ at a point which is near the begining of x and the. Newton's backward interpolation formula is useful to find the value of y at a point which is near the ending of x.

The central difference interpolation formula which are most suited for interpolation nearer to the middle of the tabulated set

Gauss Forward Interpolation Formula :—

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_{-1} + \frac{P(P-1)(P+1)}{3!}\Delta^3 y_{-1} + \frac{P(P-1)(P+2)(P+1)}{4!}\Delta^4 y_{-2} + \dots$$

Where $P = \dfrac{x - x_0}{h}$

Gauss Forward formula is used to interpolate the values of the function for the value of P such that $0 < P < 1$.

Table :-

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

For central difference formula, the central ordinate is taken as y_0 corresponding $x = x_0$.

11) Find the polynomial which fits the data in the following table using Gauss forward formula.

Sol:

x	3	5	7	9	11
y	6	24	58	108	174

Sol. The Difference table is

x	y	Δ	Δ^2	Δ^3
$3 = x_{-2}$	$6 = y_{-2}$			
		$18 = \Delta y_{-2}$		
$5 = x_{-1}$	$24 = y_{-1}$		$16 = \Delta^2 y_{-2}$	
		$34 = \Delta y_{-1}$		$0 = \Delta^3 y_{-2}$
$7 = x_0$	$58 = y_0$		$16 = \Delta^2 y_{-1}$	
		$50 = \Delta y_0$		$0 = \Delta^3 y_{-1}$
$9 = x_1$	$108 = y_1$		$16 = \Delta^2 y_0$	
		$66 = \Delta y_1$		
$11 = x_2$	$174 = y_2$			

Gauss forward interpolation formula is.

$$y = y_0 + p\Delta y_0 + \frac{(p-1)p}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-1}$$

$x_0 = 7$, $y_0 = 58$, $\Delta y_0 = 50$ $\Delta^2 y_{-1} = 16$ $\Delta^3 y_{-1} = 0$.

$$p = \frac{x - x_0}{h} = \frac{x-7}{2}$$

$$y = 58 + \left(\frac{x-7}{\cancel{2}}\right) \cdot \overset{25}{\cancel{50}} + \frac{1}{2}\left(\frac{x-7}{\cancel{2}}\right)\left(\frac{x-9}{\cancel{2}}\right) \cdot \overset{2}{\cancel{16}}$$

$$y = 58 + 25x - 175 + 2x^2 - 32x + 126$$

$$y = 2x^2 - 7x + 9.$$

(2) Find y_{30} using Gauss forward interpolation formula given (2) that

$y_{21} = 18.4708$ $y_{25} = 17.8144$ $y_{29} = 17.1070$ $y_{33} = 16.3432$ $y_{37} = 15.5154$ (14)

sol: The Difference Table is

x	y	Δ	Δ^2	Δ^3	Δ^4
$21 = x_{-2}$	$18.4708 = y_{-2}$				
		$-0.6564 = \Delta y_{-2}$			
$25 = x_{-1}$	$17.8144 = y_{-1}$		$-0.0510 = \Delta^2 y_{-2}$		
		$-0.7074 = \Delta y_{-1}$		$-0.0054 = \Delta^3 y_{-2}$	
$29 = x_0$	$17.1070 = y_0$		$-0.0564 = \Delta^2 y_{-1}$		$-0.0022 = \Delta^4 y_{-2}$
		$-0.7638 = \Delta y_0$		$-0.0076 = \Delta^3 y_{-1}$	
$33 = x_1$	$16.3432 = y_1$		$-0.064 = \Delta^2 y_0$		
		$-0.8278 = \Delta y_1$			
$37 = x_2$	$15.5154 = y_2$				

The Gauss forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{(P-1)P}{2!}\Delta^2 y_{-1} + \frac{(P-1)P(P+1)}{3!}\Delta^3 y_{-1} + \frac{(P-2)(P-1)P(P+1)}{4!}\Delta^4 y_{-2}$$

where $P = \frac{x - x_0}{h}$.

$x_0 = 29$, $y_0 = 17.1070$ $\Delta y_0 = -0.7638$ $\Delta^2 y_{-1} = -0.0564$,

$\Delta^3 y_{-1} = -0.0076$ $\Delta^4 y_{-2} = -0.0022$

$x = 30$, $h = 4$ $P = \frac{30-29}{4} = \frac{1}{4} = 0.25$

$$y = 17.1070 + (0.25)(-0.7638) + \frac{(0.25-1)(0.25)}{2!}(-0.0564) + \frac{(0.25-1)(0.25)(0.25+1)}{3!}(-0.0076)$$
$$+ \frac{(0.25-2)(0.25-1)(0.25)(0.25+1)}{4!}(-0.0022)$$

$$y = 17.1070 - 0.19095 + \frac{(-0.75)(0.25)}{2!}(-0.0564) + \frac{(-0.75)(0.25)(1.25)}{3!}(-0.0076)$$
$$+ \frac{(-0.75)(-1.75)(0.25)(1.25)}{4!}(-0.0022)$$

$= 17.1070 - 0.1905 + 0.0053 + 0.000296875 - 0.000037598$

$y = 16.9817$

(3) Given that $f(2) = 10$, $f(1) = 8$, $f(0) = 5$, $f(-1) = 10$ find $f(\frac{1}{2})$ using Gauss forward interpolation formula.

Sol. The Difference Table is.

x	y	Δ	Δ^2	Δ^3
$-1 = x_{-1}$	$10 = y_{-1}$			
		$-5 = \Delta y_{-1}$		
$0 = x_0$	$5 = y_0$		$8 = \Delta^2 y_{-1}$	
		$3 = \Delta y_0$		$-9 = \Delta^3 y_{-1}$
$1 = x_1$	$8 = y_1$		$-1 = \Delta^2 y_0$	
		$2 = \Delta y_1$		
$2 = x_2$	$10 = y_2$			

$x_0 = 0 \quad y_0 = 5 \quad \Delta y_0 = 3 \quad \Delta^2 y_{-1} = 8 \quad \Delta^3 y_{-1} = -9. \quad h = 1$

The Gauss forward interpolation formula is

$$y = y_0 + P\Delta y_0 + \frac{(P-1)P}{2!}\Delta^2 y_{-1} + \frac{(P-1)P(P+1)}{3!}\Delta^3 y_{-1}.$$

where $P = \frac{x - x_0}{h} = \frac{\frac{1}{2} - 0}{1} = \frac{1}{2}$

$$y = 5 + \frac{1}{2}(3) + \frac{-1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}(8) + \frac{-1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{3!}(-9)$$

$y = 5 + 1.5 - 1 + 0.5625$

$y = 6.0625$

GAUSS FORWARD INTERPOLATION FORMULA

(1) Find the polynomial which fits the data in the following table using Gauss forward interpolation formula.

x	3	5	7	9	11
y	6	24	58	108	174

Ans:- $2x^2-7x+9$.

(2) Use the Gauss forward interpolation formula to find $f(3.3)$ from the following table.

x	1	2	3	4	5
$y=f(x)$	15.30	15.10	15.00	14.50	14.00

Ans:- 14.8912.

(3) Using the Gauss forward interpolation formula, find the value of $\log_{10}^{347.5}$ from the following table.

x	320	330	340	350	360
$y=\log_{10}^{x}$	2.5052	2.5185	2.5315	2.5441	2.5563

Ans:- 2.54099.

(4) Use Gauss forward interpolation formula to find y_{30} given $y_{21}=18.4708$ $y_{25}=17.8144$ $y_{29}=17.1070$, $y_{33}=16.3432$ $y_{37}=15.5154$.

Ans:- 16.9213.

(5) Find the value of e^{-x} when $x=1.725$, 1.7489 and $x=1.775$ from the following table, using the Suitable interpolation formulas.

x	1.72	1.73	1.74	1.75	1.76	1.77	1.78
e^{-x}	0.179066	0.177284	0.175520	0.173774	0.172045	0.170333	0.168638

Ans:- , 0.173965 ,

(6) From the following table for the function $y=e^x$, find e^x for $x=1.75$, 1.91 and 2.15 using an appropriate interpolation formula.

x	1.7	1.8	1.9	2.0	2.1	2.2
$y=e^x$	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

Ans:- , 6.7531 ,

Gauss Backward Interpolation Formula :—

$$y = y_0 + p\,\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-2} + \frac{(p-1)p(p+1)(p+2)}{4!}\Delta^4 y_{-2} + \frac{(p-2)(p-1)p(p+1)(p+2)}{5!}\Delta^5 y_{-3} + \cdots$$

where $p = \dfrac{x - x_0}{h}$

The Gauss Backward formula is used to interpolate the values for the value of p such that $-1 < p < 0$.

Table :-

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

For central difference formula, the central ordinate is taken as y_0 corresponding $x = x_0$.

(1) Use Gauss Backward interpolation formula to find the value of y at $x = 1936$ using the following table.

x	1901	1911	1921	1931	1941	1951.
y	12	15	20	27	39	52

sol:- The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_{-4} 1901	$12 = y_{-4}$					
		$3 = \Delta y_{-4}$				
x_{-3} 1911	$15 = y_{-3}$		$2\ \Delta^2 y_{-4}$			
		$5\ \Delta y_{-3}$		$0\ \Delta^3 y_{-4}$		
x_{-2} 1921	$20 = y_{-2}$		$2\ \Delta^2 y_{-3}$		$3\ \Delta^4 y_{-4}$	
		$7\ \Delta y_{-2}$		$3\ \Delta^3 y_{-3}$		$\Delta^5 y_{-4}$ -10.
x_{-1} 1931	$27 = y_{-1}$		$5\ \Delta^2 y_{-2}$		$-7\ \Delta^4 y_{-3}$	
		$12\ \Delta y_{-1}$		$-4\ \Delta^3 y_{-2}$		
x_0 1941	$39 = y_0$		$1\ \Delta^2 y_{-1}$			
		$13\ \Delta y_0$				
x_1 1951	$52 = y_1$					

The Gauss Backward interpolation formula is

$$y = y_0 + p\,\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-2}.$$

where $p = \dfrac{x - x_0}{h}$

$x_0 = 1941 \quad y_0 = 39 \quad \Delta y_{-1} = 12 \quad \Delta^2 y_{-1} = 1 \quad \Delta^3 y_{-2} = -4$

$x = 1936 \quad p = \dfrac{x - x_0}{h} = \dfrac{1936 - 1941}{10} = \dfrac{-5}{10} = -0.5$

$h = 10$

$$y = 39 + (-0.5)\,12 + \frac{(-0.5)(-0.5+1)}{2!}(1) + \frac{(-0.5-1)(-0.5)(-0.5+1)}{3!}(-4)$$

$$y = 39 - 6 + 0.125 - 0.25$$

$$y = 32.625$$

(2) Use Gauss Backward interpolation formula find $y(8)$ form the following table.

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Sol:- The difference table is.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0 x_{-2}	7 y_{-2}					
		$4\ \Delta y_{-2}$				
x_{-1} 5	$11 = y_{-1}$		$-1\ \Delta^2 y_{-2}$			
		$3\ \Delta y_{-1}$		$2\ \Delta^3 y_{-2}$		
$x_0 = 10$	$y_0 = 14$		$1\ \Delta^2 y_{-1}$		$-1\ \Delta^4 y_{-2}$	
		$4\ \Delta y_0$		$1\ \Delta^3 y_{-1}$		$0\ \Delta^5 y_{-2}$
x_1 15	18 y_1		$2\ \Delta^2 y_0$		$-1\ \Delta^4 y_{-1}$	
		$6\ \Delta y_1$		$0\ \Delta^3 y_0$		
20 x_2	24 y_2		$2\ \Delta^2 y_1$			
		$8\ \Delta y_2$				
25 x_3	32 y_3					

The Gauss Backward interpolation formula is

$$y = y_0 + p\,\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p-1)p(p+1)}{3!}\Delta^3 y_{-2} + \frac{(p-1)p(p+1)(p+2)}{4!}\Delta^4 y_{-2}$$

Where $p = \dfrac{x - x_0}{h}$

$x_0 = 10 \quad y_0 = 14 \quad \Delta y_{-1} = 3 \quad \Delta^2 y_{-1} = 1 \quad \Delta^3 y_{-2} = 2$

$\Delta^4 y_{-2} = -1 \quad h = 5 \quad x = 8$

$$p = \frac{x - x_0}{h} = \frac{8-10}{5} = \frac{-2}{5} = -0.4$$

$$y = 14 + (-0.4)(3) + \frac{(-0.4)(-0.4+1)}{2!}(1) + \frac{(-0.4)(-0.4+1)(-0.4-1)}{3!} 2$$

$$+ \frac{(-0.4+2)(-0.4)(-0.4+1)(-0.4-1)}{4!}(-1)$$

$$y = 14. -1.2 + \frac{(-0.4)(0.6)}{2} + \frac{(0.4)(0.6)(-1.4)}{3!} \cdot 2$$

$$+ \frac{(1.6)(-0.4)(0.6)(-1.4)}{4!}(-1)$$

$$y = 14 - 1.2 - 0.12 + 0.112 - 0.0224$$

$$y = \cancel{12.2} \; 12.7696.$$

Dis advantages of backward difference interpolation :-

(i) In Gauss Backward interpolation we can not find polynomia for the given data.

(ii) Newton Backward interpolation does not gives exact accurac for central values.

GAUSS BACKWARD INTERPOLATION FORMULA.

(1) From the following table for the function $y = f(x)$, find y at $x = 32$ using the Gauss backward interpolation formula.

x	25	30	35	40
$y = f(x)$	0.2707	0.3027	0.3386	0.3794

Ans:- 0.3165.

(2) Given that $\sqrt{6500} = 80.6226$ $\sqrt{6510} = 80.6846$ $\sqrt{6520} = 80.7466$. $\sqrt{6530} = 80.8084$. find $\sqrt{6516}$ by using the Gauss backward interpolation formula. Ans:- 80.7218.

(3) Apply Gauss backward interpolation formula to find y when $x = 25$ from the following table.

x	20	24	28	32
y	2854	3162	3544	3992

Ans:- 3251.25.

(4) For the following data estimate $f(1.720)$, $f(2.68)$ and $f(2.36)$ using an appropriate difference formulae.

x	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
y	0.0495	0.0605	0.0739	0.0903	0.1102	0.1346	0.1644	0.2009.

Ans:-

(5) From the following table for the function $y = f(x)$, find y when $x = 1.35$, $x = 1.15$ and $x = 1.9$ using an appropriate difference formulae.

x	1	1.2	1.4	1.6	1.8	2
y	0.0	−0.112	−0.016	0.336	0.992	2.0

Ans:- , −0.062,

(6) The following table gives the population y (in lakhs) of a certain city in the year x. Find by using an appropriate formula the population in the years 1965, 1985 and 2005.

x	1960	1970	1980	1990	2000	2010
y	12	15	20	27	39	52.

Stirling's Formula :—

(17) (5)

Gauss forward difference formula is given by

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \quad (1)$$

Gauss backward difference formula is given by

$$y = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \quad (2)$$

Taking the average of (1) and (2), we will get Stirling's formula.

$$\frac{y+y}{2} = \frac{y_0+y_0}{2} + \frac{p\Delta y_0 + p\Delta y_{-1}}{2} + \frac{1}{2}\left[\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1}\right]$$

$$+\frac{1}{2}\left[\frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}\right] +$$

$$\frac{1}{2}\left[\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2}\right] + \cdots$$

$$y = y_0 + \frac{p}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{p}{2}\left[\frac{p-1}{2} + \frac{p+1}{2}\right]\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{2.3!}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right]$$

$$+\frac{1}{2}\cdot\frac{p(p-1)(p+1)}{4!}\left[p-2+p+2\right]\Delta^4 y_{-2} + \cdots$$

$$y = y_0 + \frac{p}{2}\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

This is Stirling's formula, it gives the most accurate result for $-0.25 \le p \le 0.25$.

Therefore we have to choose x_0 such that p satisfies this inequality.

(1) Use stirling's formula to find y_{32} from the following table.

$y_{20} = 14.035$, $y_{25} = 13.674$ $y_{30} = 13.257$, $y_{35} = 12.734$

$y_{40} = 12.089$ $y_{45} = 11.309$.

Sol: The forward difference table is.

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_{-2} 20	14.035 y_{-2}					
		Δy_{-2} -0.361				
x_{-1} 25	13.674 y_{-1}		$\Delta^2 y_{-2}$ -0.056			
		Δy_{-1} -0.417		$\Delta^3 y_{-2}$ -0.05		
$x_0 = 30$	13.257 y_0		$\Delta^2 y_{-1}$ -0.106		$\Delta^4 y_{-2}$ 0.034	
		Δy_0 -0.523		$\Delta^3 y_{-1}$ -0.016		$\Delta^5 y_{-2}$ -0.031
x_1 35	12.734 y_1		$\Delta^2 y_0$ -0.122		$\Delta^4 y_{-1}$ 0.003	
		Δy_1 -0.645		$\Delta^3 y_0$ -0.013		
x_2 40	12.089 y_2		$\Delta^2 y_1$ -0.135			
		Δy_2 -0.78				
x_3 45	11.309 y_3					

The stirling's formula is given by

$$y = y_0 + \frac{p}{2}\left(\Delta y_0 + \Delta y_{-1}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$

$$\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Here $x = 32$ $x_0 = 30$ $h = 5$ $p = \frac{x - x_0}{h}$

$$p = \frac{x - x_0}{h} = \frac{32-30}{5} = 0.4.$$

$y_0 = 13.257$ $\Delta y_0 = -0.523$ $\Delta y_{-1} = -0.417$.

$\Delta^2 y_{-1} = -0.106$ $\Delta^3 y_{-2} = -0.05$ $\Delta^3 y_{-1} = -0.016$

$\Delta^4 y_{-2} = -0.034$ $\Delta^4 y_{-1} = 0.003$.

$$y = 13.257 + \frac{0.4}{2}(-0.523 - 0.417) + \frac{(0.4)^2}{2}(-0.106)$$

$$+ \frac{(0.4)((0.4)^2 - 1)}{3!}\left[\frac{-0.05 - 0.016}{2}\right] + \frac{(0.4)^2((0.4)^2 - 1)}{4!}(-0.034)$$

$$y = 13.257 - 0.188 - 0.00848 + 0.001848 - 0.0001904$$

$$y = 13.062$$

(2) Use stirling's formula to find y_{28}, given that $y_{20} = 49225$

$y_{25} = 48316$, $y_{30} = 47236$, $y_{35} = 45926$, $y_{40} = 44306$.

Ans :- 47691.82

STIRLING'S FORMULA.

(1) Using the stirling's formula, find $f(31)$ from the following table

x	20	25	30	35	40
$f(x)$	49.225	48.316	47.236	45.296	44.306

Ans:- 46.895

(2) Apply stirling's formula to find $y_{13.8}$ from the following table.

x	10	12	14	16	18
y_x	0.240	0.281	0.318	0.352	0.384.

Ans:- 0.31445

(3) From the following table for the function $y = \tan x$, find $\tan 14'$ and $\tan 16'$ using the stirling's formula.

x	0	5	10	15	20	25
$y=\tan x$	0	0.0875	0.1763	0.2679	0.3640	0.4663.

Ans:- 0.2493, 0.2867.

(4) Apply stirling's formula to find $\log_{10} 337.5$ from the following table.

x	310	320	330	340	350	360
$\log_{10} x$	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

Ans:- 2.52828.

(5) Given $\sqrt{1} = 1$, $\sqrt{1.05} = 1.0247$ $\sqrt{1.10} = 1.0488$ $\sqrt{1.15} = 1.0724$ $\sqrt{1.20} = 1.0954$ $\sqrt{1.25} = 1.1180$ $\sqrt{1.30} = 1.1402$ find $\sqrt{1.12}$ using the stirling's formula.

Ans:- 1.0583.

(6) Employ stirling's formula to find $y_{11.8}$ and $y_{12.2}$ from the following table

x	10	11	12	13	14
y_x	0.23967	0.28060	0.31788	0.35209	0.38368

Ans:- , 0.32497.

Interpolation with unevenly spaced points :-

Lagrange's Interpolation Formula :-

Let $x_0, x_1, x_2 \dots x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2 \dots y_n$ be the corresponding values of $y = f(x)$. Let the polynomial of degree n for the function $y = f(x)$ passing through the $(n+1)$ points $(x_0, f(x_0))$ $(x_1, f(x_1))$ $(x_2, f(x_2)) \dots (x_n, f(x_n))$

The Lagrange's interpolation formula is given by

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) + \dots$$
$$+ \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} f(x_n).$$

Find the unique polynomial $P(x)$ of degree 2 or less such that $P(1) = 1$ $P(3) = 27$ $P(4) = 64$ using Lagrange interpolation formula.

Sol:- Given that $x_0 = 1$ $x_1 = 3$ $x_2 = 4$

$y_0 = 1$ $y_1 = 27$ $y_2 = 64$.

The Lagrange's Interpolation formula is given by.

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

$$= \frac{(x-3)(x-4)}{(1-3)(1-4)} \cdot 1 + \frac{(x-1)(x-4)}{(3-1)(3-4)} \cdot 27 + \frac{(x-1)(x-3)}{(4-1)(4-3)} \cdot 64$$

$$= \frac{1}{6}\left[48x^2 - 114x + 72\right]$$

$$= 8x^2 - 19x + 12$$

Using Lagrange's interpolation formula, find the form of the fun. f(x) from the following table.

x	0	1	3	4
$f(x)$	-12	0	12	24

Sol:- Here $x_0 = 0 \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 4$

$f(x_0) = -12 \quad f(x_1) = 0 \quad f(x_2) = 12, \quad f(x_3) = 24.$

The values of x are unequally spaced so we apply Lagranges interpolation formula.

Lagranges interpolation formula is given by

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3).$$

$$y = f(x) = \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) + \frac{}{} 0$$

$$+ \frac{(x-0)(x-1)(x-4)}{(3)(2)(-1)} 12 + \frac{(x-0)(x-1)(x-3)}{(4)(3)(1)} 24.$$

$$y = f(x) = (x-1)(x-3)(x-4) + 12x(x-1)(x-4) + 2x(x-1)(x-3)$$

$$\therefore \quad y = x^3 - 6x^2 + 17x - 12$$

Using Lagrange's interpolation formula, express $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as sum of partial fractions.

Sol: Let $f(x) = 3x^2+x+1$

Take $(x-1)(x-2)(x-3) = 0$

$x = 1, 2, 3$

$x_0 = 1 \quad x_1 = 2 \quad x_2 = 3$

$f(x_0) = 5 \quad f(x_1) = 15 \quad f(x_2) = 31.$

By Lagrange's interpolation formula.

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$3x^2+x+1 = \frac{(x-2)(x-3)}{(-1)(-2)} \cdot 5 + \frac{(x-1)(x-3)}{1.(-1)} 15 + \frac{(x-1)(x-2)}{2.1} (31)$$

Divide with $(x-1)(x-2)(x-3)$, we get

$$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}.$$

LAGRANGE'S INTERPOLATION FORMULA.

(1) Find the interpolating polynomial for the data given in the following table.

x	0	1	4	5
y	4	3	24	39

Ans:- $2x^3 - 3x + 4$.

(2) Using Lagrange's interpolation formula, fits a polynomial to the following data. Hence find $y(1)$

x	-1	0	2	3
y	-8	3	1	12

Ans:- $2x^3 - 6x^2 + 3x + 3$, Ans:- $y(1) = 2$.

(3) Find the parabola passing through the points (0, 1) (1, 3) and (3, 55). using the Lagrange's interpolation formula.

Ans:- $8x^2 - 6x + 1$.

(4) A curve passes through the points (0, 18) (1, 10) (3, −18) and (6, 90) Find the slope of the curve at $x = 2$.

Ans:- $y = 2x^3 - 10x^2 + 18$, $y'(2) = -16$.

(5) obtain the third degree polynomial passing through the four points given below. Hence estimate $f(2.5)$ and $\int_1^2 f(x)dx$.

x	1	1.5	2.0	2.8
$f(x)$	3	3.375	5	12.0172

(6) Find $y(5)$, given that $y(0) = 1$ $y(1) = 3$ $y(3) = 13$ $y(8) = 123$ using Lagrange's formula.

(7) Using Lagrange's interpolation formula, find y when $x = 10$.

x	2	3	8	14
y	94.8	87.9	81.3	68.7

Ans:- 74.925

(8) Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$ $f(5) = -248$ $f(6) = 0$ $f(9) = 13104$. find $f(2)$ using Lagrange's interpolation formula. Ans:- 28.

~~Inverse~~ Interpolation :—

So far, given a set of values of x and y, we have been finding the values of y corresponding to a certain value of x. On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called the inverse interpolation.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable.

Therefore, on interchanging x and y in the Lagrange's formula, we obtain.

$$x = f(y) = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}x_1 + \cdots$$

$$+ \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}x_n .$$

(1) Find the value of x when $y = 0.3$ by applying Lagrange's formula inversely.

x	0.4	0.6	0.8
y	0.3683	0.3332	0.2897

Sol: Here $x_0 = 0.4$ $\quad x_1 = 0.6$ $\quad x_2 = 0.8$

$y_0 = 0.3683$ $\quad y_1 = 0.3332$ $\quad y_2 = 0.2897$.

$y = 0.3$.

Inverse Lagranges interpolation formula is.

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

Sub. the given values in above formula, we get

$$x = \frac{(0.3-0.3332)(0.3-0.2897)}{(0.3683-0.3332)(0.3683-0.2897)}(0.4) + \frac{(0.3-0.3683)(0.3-0.2897)}{(0.3332-0.3683)(0.3332-0.2897)}(0.6) + \frac{(0.3-0.3683)(0.3-0.3332)}{(0.2897-0.3683)(0.2897-0.3332)}(0.8)$$

$$x = -0.0495 + 0.2764 + 0.5305$$

$$x = 0.7574.$$

INVERSE LAGRANGE'S INTERPOLATION.

(1) Given that $f(0) = 16.35$ $f(5) = 14.88$ $f(10) = 13.59$ $f(15) = 12.46$ find x when $f(x) = 14$.

Ans:- 8.33686.

(2) Given that $x_0 = 3$, $x_1 = 5$ $x_2 = 7$ $x_3 = 9$ $x_4 = 11$ $y_0 = 6$ $y_1 = 24$ $y_2 = 58$ $y_3 = 108$ $y_4 = 174$ Find x when $y = 100$.

Ans:- 8.65471

(3) Use Lagranges formula inversly to obtain the value of x when $y = 85$ from the following table.

x	2	5	8	14
y	94.8	87.9	81.3	68.7

Ans:- 6.30383

(4) Find x when $f(x) = 163$ from the following table. Ans:- 82.8

x	80	82	84	86	88
$f(x)$	134	154	176	200	221

(5) Find the value of x when $y = 13.6$ from the following table.

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Ans:- 43.1.

(6) Given that $f(10) = 1754$ $f(15) = 2648$ $f(20) = 3564$ find x when $f(x) = 3000$

Ans:- 16.9.

(7) Find the value of x when $y = 0.3$ by applying Lagrange's formula inversly.

x	0.4	0.6	0.8
y	0.3683	0.3332	0.2897.

Divided Differences :—

(22) (10)

In the construction of finite difference tables, x is assumed to be equally spaced. If x is not equally spaced, Lagrange's formula is used to find the unknown value from the table. If an another interpolation point is added to the tabulated data, then the Lagrangian coefficients are to be recalculated which results a different Lagrange's polynomial of higher degree. This difficulty will overcome by taking the Newton's divided differences.

Newton's Divided difference formula :—

Let $y = f(x)$ be a function.

Let $f(x_0), f(x_1), f(x_2) \ldots f(x_n)$ be the values corresponding to $x = x_0, x_1, x_2, \ldots x_n$. Where the values of x are not equally spaced.

The Newton's divided difference formula is

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1}) f(x_0,x_1,\ldots x_n)$$

Newton's divided difference table is .

x	$y=f(x)$	Δ	Δ^2	Δ^3
x_0	$f(x_0)$			
		$f(x_0,x_1)=\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$		
			$f(x_0,x_1,x_2)=\dfrac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$	
x_1	$f(x_1)$			$f(x_0,x_1,x_2,x_3)=\dfrac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{x_3-x_0}$
		$f(x_1,x_2)=\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$		
			$f(x_1,x_2,x_3)=\dfrac{f(x_2,x_3)-f(x_1,x_2)}{x_3-x_1}$	$f(x_1,x_2,x_3,x_4)=\dfrac{f(x_2,x_3,x_4)-f(x_1,x_2,x_3)}{x_4-x_1}$
x_2	$f(x_2)$			
		$f(x_2,x_3)=\dfrac{f(x_3)-f(x_2)}{x_3-x_2}$		
			$f(x_2,x_3,x_4)=\dfrac{f(x_3,x_4)-f(x_2,x_3)}{x_4-x_2}$	$f(x_2,x_3,x_4,x_5)=\dfrac{f(x_3,x_4,x_5)-f(x_2,x_3,x_4)}{x_5-x_2}$
x_3	$f(x_3)$			
		$f(x_3,x_4)=\dfrac{f(x_4)-f(x_3)}{x_4-x_3}$		
			$f(x_3,x_4,x_5)=\dfrac{f(x_4,x_5)-f(x_3,x_4)}{x_5-x_3}$	$f(x_3,x_4,x_5,x_6)=\dfrac{f(x_4,x_5,x_6)-f(x_3,x_4,x_5)}{x_6-x_3}$
x_4	$f(x_4)$			
		$f(x_4,x_5)=\dfrac{f(x_5)-f(x_4)}{x_5-x_4}$		
			$f(x_4,x_5,x_6)=\dfrac{f(x_5,x_6)-f(x_4,x_5)}{x_6-x_4}$	
x_5	$f(x_5)$			
		$f(x_5,x_6)=\dfrac{f(x_6)-f(x_5)}{x_6-x_5}$		
x_6	$f(x_6)$			

Compute $f(3)$ using Newton's divided difference formula from the following table (11)

x	1	2	4	8	10
$f(x)$	0	1	5	21	27

Sol:- $x_0 = 1 \quad x_1 = 2 \quad x_2 = 4 \quad x_3 = 8 \quad x_4 = 10$

$f(x_0) = 0 \quad f(x_1) = 1 \quad f(x_2) = 5 \quad f(x_3) = 21 \quad f(x_4) = 27.$

Here the values of x are un equally spaced.

so we apply Newton's divided difference formula.

Newton's divided difference formula is

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4).$$

The divided difference table is

x	$f(x)$	Δ	Δ^2		
1	0				
		$\frac{1-0}{2-1} = 1$			
			$\frac{2-1}{4-1} = \frac{1}{3}$		
2	1				
				$\frac{\frac{1}{3}-\frac{1}{3}}{8-1} = 0$	
		$\frac{5-1}{4-2} = 2$			
					$\frac{-\frac{1}{16}-0}{10-1} = \frac{-1}{144}$
			$\frac{4-2}{8-2} = \frac{1}{3}$		
4	5			$\frac{-\frac{1}{6}-\frac{1}{3}}{10-2} = \frac{-1}{16}$	
		$\frac{21-5}{8-4} = 4$			
			$\frac{3-4}{10-4} = \frac{-1}{6}$		
8	21				
		$\frac{27-21}{10-8} = 3$			
10	27				

$f(x_0, x_1) = 1$

$f(x_0, x_1, x_2) = \frac{1}{3}$ $\quad f(x_0, x_1, x_2, x_3) = 0$ $\quad f(x_0, x_1, x_2, x_3, x_4) = \frac{-1}{144}$

sub. all these values in Newton's divided diff. formula, we get

$$f(x) = 0 + (x-1).1 + (x-1)(x-2).\frac{1}{3} + (x-1)(x-2)(x-4).0 +$$

$$(x-1)(x-2)(x-4)(x-8).\frac{-1}{144}$$

$$f(x) = \frac{x^2-1}{3} - \frac{1}{144}(x^4 - 15x^3 + 70x^2 - 120x + 64)$$

At $x=3$, $f(3) = \frac{3^2-1}{3} - \frac{1}{144}(3^4 - 15.3^3 + 70.3^2 - 360 + 64)$.

$$= 2.5972.$$

NEWTONS DIVIDED DIFFERENCE FORMULA

1. Given the values.

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Ans:- 810.

Evaluate $f(9)$, using Newton's divided difference formula.

2. Determine $f(x)$ as a polynomial in x for the following data.

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Ans:- $3x^4 - 5x^3 + 6x^2 - 14x + 5$

3. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$. Ans:- 1

4. Use Newton's divided difference method to compute $f(5.5)$ from the following data.

x	0	1	4	5	6
$f(x)$	1	14	15	6	3

Ans:- 3.09

5. Using Newtons divided difference formula Evaluate $f(8)$ and $f(15)$ given

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Ans:- 448, 3150.

6. obtain the Newton's divided difference interpolation polynomial and hence find $f(6)$.

x	3	7	9	10
$f(x)$	168	120	72	63

Ans:- 133.19

7. Using Newtons divided difference interpolation, find the polynomial of the given data.

x	-1	0	1	3
$f(x)$	2	1	0	-1

Ans:- $f(x) = \frac{1}{24}x^3 - \frac{1}{6}x^2 - 25x + \frac{557}{60}x - 1$

Symbolic Relations and separation of Symbols :-

Forward difference operator :- The forward difference operator is denoted by Δ and is defined as $\Delta f(x) = f(x+h) - f(x)$.

Where h is the increment in x.

Backward difference operator :- The backward difference operator is denoted by ∇ and is defined as $\nabla f(x) = f(x) - f(x-h)$.

Central difference operator :- The central difference operator is denoted by δ and is defined as

$$\delta y_{\frac{1}{2}} = y_1 - y_0$$

$$\delta y_{\frac{3}{2}} = y_2 - y_1$$

$$\delta y_{\frac{5}{2}} = y_3 - y_2$$

$$\cdots\cdots$$

$$\delta y_{\frac{2n-1}{2}} = y_n - y_{n-1}$$

Mean or Average operator :-

The mean or average operator μ is defined as $\mu y_r = \frac{1}{2}\left[y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right]$

The shift Operator :-

The shift operator E is defined by the equation $Ey_r = y_{r+1}$.

This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}.

A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$.

In general $E^n y_r = y_{r+n}$

We have $\Delta f(x) = f(x+h) - f(x)$

$$f(x+h) = f(x) + \Delta f(x) = (1+\Delta) f(x)$$

This shows that the operator $1+\Delta$ operating on $f(x)$ shifts $f(x)$ forward to its immediately succeeding value $f(x+h)$ ②

We denote this operator by E and refer to it as the (first order) shift operator.

Thus by definition $E = 1+\Delta$

and we have $Ef(x) = f(x+h)$. $\left[\because Ef(x) = (1+\Delta)f(x)\right]$.

$$E^2 f(x) = E\{Ef(x)\} = E\{f(x+h)\} = f(x+2h).$$

$$E^3 f(x) = E\{E^2 f(x)\} = E\{f(x+2h)\} = f(x+3h).$$

...

In general $E^n f(x)$ is defined by $E^n f(x) = f(x+nh), \; n = 1,2,3 \ldots$ ①

The operator E^n shifts the value of a function at x to its value at $x+nh$. This operator is referred to as the n^{th} order shift operator.

The formula ① can be put in the following alternative form.

$$E^n y_r = y_{r+n}$$

<u>The inverse shift operator</u> :-

Inverse operator E^{-1} is defined as $E^{-1} y_r = y_{r-1}$.

In general $E^{-n} y_r = y_{r-n}$.

Since $\Delta f(x) = \nabla f(x+h)$

We have $\nabla f(x) = \Delta f(x-h) = f(x) - f(x-h)$

$$f(x-h) = f(x) - \nabla f(x)$$

$$= (1-\nabla) f(x).$$

Thus, the operator $(1-\nabla)$ operating on $f(x)$ shifts $f(x)$ backward to its immediately preceding value $f(x-h)$. We denote this operator by E^{-1} (or $\frac{1}{E}$) and refer to it as the (first-order) inverse shift operator. Thus $E^{-1} = 1-\nabla$. 3

and we have $E^{-1}f(x) = f(x-h)$.

$$E\left[E^{-1}f(x)\right] = E\,f(x-h) = f(x)$$

$$E^{-1}\left[E\,f(x)\right] = E^{-1}\left[f(x+h)\right] = f(x).$$

This means that the operator E^{-1} is actually the inverse of the operator E.

The nth order inverse shift operator E^{-n} is defined by an expression

$$E^{-n}f(x) = f(x-nh) \quad , \quad n = 1, 2, 3, \ldots \quad \text{①}$$

Thus, the operator E^{-n} shifts the value of a function at x to its value at $x-nh$.

The formula ① can be put in the following alternative form.

$$E^{-n}y_r = y_{r-n}$$

Relation between the Operators :-

(i) $\Delta = E-1$

We have $\Delta y_0 = y_1 - y_0$.

$$\Delta y_0 = E y_0 - y_0 \qquad [\because E^n y_r = y_{r+n}]$$

$$\Delta y_0 = (E-1)y_0$$

$$\Delta = E-1.$$

(OR)

$$\Delta f(x) = f(x+h) - f(x) \qquad \because E f(x) = f(x+h)$$

$$= E f(x) - f(x) = (E-1) f(x)$$

$$\therefore \Delta = E-1$$

(ii) $\nabla = 1 - E^{-1}$

We have. $\nabla y_1 = y_1 - y_0$

$= y_1 - E^{-1} y_1$ $\quad \left[\because E^{-n} y_r = y_{r-n}\right]$

$\nabla y_1 = (1 - E^{-1}) y_1$

$\nabla = (1 - E^{-1})$

(OR)

$\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$ $\quad \because E^{-1} f(x) = f(x-h)$

$\nabla f(x) = (1 - E^{-1}) f(x)$

$\nabla = 1 - E^{-1}.$

(iii) $\delta = E^{1/2} - E^{-1/2}$

We have $\delta y_{\frac{1}{2}} = y_1 - y_0$

$= E^{1/2} y_{1/2} - E^{-1/2} y_{1/2}$ $\quad \left[\because E^{n} y_r = y_{r+n} \quad E^{-n} y_r = y_{r-n}\right]$

$\delta y_{\frac{1}{2}} = (E^{1/2} - E^{-1/2}) y_{1/2}$

$\delta = E^{1/2} - E^{-1/2}$

(iv) $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

We have $\mu y_r = \frac{1}{2}\left[y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right]$

$= \frac{1}{2}\left[E^{1/2} y_r + E^{-1/2} y_r\right]$

$\mu y_r = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] y_r$

$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right].$

Prove the following (a) $y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0$. (6)

(b) $y_3 = y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0$.

Sol: (a) We have $\Delta y_0 = y_1 - y_0 \implies y_1 = y_0 + \Delta y_0$.

$$\Delta y_1 = y_2 - y_1 \implies y_2 = y_1 + \Delta y_1.$$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0.$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0. \implies \Delta y_1 = \Delta y_0 + \Delta^2 y_0.$$

$$y_2 = y_1 + \Delta y_1$$

$$= (y_0 + \Delta y_0) + (\Delta y_0 + \Delta^2 y_0)$$

$$\therefore y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0$$

(b) $\Delta y_2 = y_3 - y_2 \implies y_3 = y_2 + \Delta y_2$.

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1.$$

$$\Delta y_2 = \Delta y_1 + \Delta^2 y_1.$$

$$\Delta^3 y_0 = \Delta^2(\Delta y_0) = \Delta^2(y_1 - y_0) = \Delta^2 y_1 - \Delta^2 y_0$$

$$\Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0.$$

$$y_3 = y_2 + \Delta y_2.$$

$$= (y_0 + 2\Delta y_0 + \Delta^2 y_0) + (\Delta y_1 + \Delta^2 y_1)$$

$$= (y_0 + 2\Delta y_0 + \Delta^2 y_0) + (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0)$$

$$y_3 = y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0.$$

(v) $\mu^2 = 1 + \frac{1}{4}\delta^2$

We have $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

$$\mu^2 = \frac{1}{4}\left(E^{1/2} + E^{-1/2}\right)^2$$

$$= \frac{1}{4}\left[\left(E^{1/2} - E^{-1/2}\right)^2 + 4E^{1/2}E^{-1/2}\right]$$

$$= \frac{1}{4}\left[\delta^2 + 4\right]$$

$$\mu^2 = 1 + \frac{1}{4}\delta^2.$$

(vi) $E = e^{hD}$

We have $E f(x) = f(x+h)$.

By Taylor series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$$= f(x) + \frac{h}{1!}\frac{d}{dx}(f(x)) + \frac{h^2}{2!}\frac{d^2}{dx^2}(f(x)) + \frac{h^3}{3!}\frac{d^3}{dx^3}(f(x)) + \cdots$$

$$= f(x) + hD(f(x)) + \frac{h^2}{2!}D^2(f(x)) + \frac{h^3}{3!}D^3(f(x)) + \cdots$$

$$f(x+h) = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right)f(x).$$

$$E f(x) = e^{hD} f(x) \qquad \left[\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right].$$

$$\therefore \quad E = e^{hD}$$

Prove the following

(i) $\Delta y_r^2 = (y_r + y_{r+1})\,\Delta y_r$

(ii) $\sum_{r=0}^{n-1} \Delta^2 y_r = \Delta y_n - \Delta y_0$

Sol: (i) $\Delta y_r^2 = y_{r+1}^2 - y_r^2$

$= (y_{r+1} + y_r)(y_{r+1} - y_r)$

$= (y_{r+1} + y_r)\,\Delta y_r$.

(ii) $\sum_{r=0}^{n-1} \Delta^2 y_r = \Delta^2 y_0 + \Delta^2 y_1 + \Delta^2 y_2 + \dots + \Delta^2 y_{n-2} + \Delta^2 y_{n-1}$

$= \Delta(\Delta y_0) + \Delta(\Delta y_1) + \Delta(\Delta y_2) + \dots + \Delta(\Delta y_{n-2}) + \Delta(\Delta y_{n-1})$

$= \Delta(y_1 - y_0) + \Delta(y_2 - y_1) + \Delta(y_3 - y_2) + \dots + \Delta(y_{n-1} - y_{n-2})$
$+ \Delta(y_n - y_{n-1})$

$= (\Delta y_1 - \Delta y_0) + (\Delta y_2 - \Delta y_1) + (\Delta y_3 - \Delta y_2) + \dots + (\Delta y_{n-1} - \Delta y_{n-2})$
$+ (\Delta y_n - \Delta y_{n-1})$

$= \Delta y_n - \Delta y_0$

Evaluate (i) $\Delta \cos x$ (ii) $\Delta \log f(x)$ (iii) $\Delta^2 \sin(px+q)$ (iv) $\Delta \tan^{-1} x$.
(v) $\Delta^n (e^{ax+b})$.

Sol: We have $\Delta f(x) = f(x+h) - f(x)$.

(i) Let $f(x) = \cos x$

$\Delta \cos x = \cos(x+h) - \cos x = -2\sin\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$.

(ii) Let $f_1(x) = \log f(x)$.

$\Delta \log f(x) = \log f(x+h) - \log f(x) = \log\left[\frac{f(x+h)}{f(x)}\right]$.

$= \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right] = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$.

(iii) Let $f(x) = \sin(px+q)$.

$$\Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q)$$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\left(\frac{ph}{2}\right)$$

$$= 2\sin\left(\frac{ph}{2}\right)\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2 \sin(px+q) = 2\sin\left(\frac{ph}{2}\right)\Delta\left[\sin\left[(px+q)+\frac{1}{2}(\pi+ph)\right]\right]$$

$$= 2\sin\left(\frac{ph}{2}\right)\left\{\sin\left[p(x+h)+q)+\frac{1}{2}(\pi+ph)\right] - \sin\left[(px+q)+\frac{1}{2}(\pi+ph)\right]\right\}$$

$$= 2\sin\left(\frac{ph}{2}\right)2\cos\left[(px+q)+\left(\frac{ph}{2}\right)+\frac{1}{2}(\pi+ph)\right]\sin\left(\frac{ph}{2}\right)$$

$$= \left[2\sin\left(\frac{ph}{2}\right)\right]^2 \sin\left[(px+q)+2\cdot\frac{1}{2}(\pi+ph)\right].$$

(iv) Let $f(x) = \tan^{-1}x$.

$$\Delta \tan^{-1}x = \tan^{-1}(x+h) - \tan^{-1}x.$$

$$= \tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)$$

$$= \tan^{-1}\left(\frac{h}{1+x(x+h)}\right)$$

(v) Let $f(x) = e^{ax+b}$

$$\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b} = e^{ax+b}(e^{ah}-1)$$

$$\Delta^2 e^{ax+b} = \Delta[\Delta(e^{ax+b})] = \Delta[e^{ax+b}(e^{ah}-1)] = (e^{ah}-1)\Delta(e^{ax+b})$$

$$= (e^{ah}-1)(e^{ah}-1)e^{ax+b} = (e^{ah}-1)^2 e^{ax+b}.$$

Proceeding like this, we get $\Delta^n(e^{ax+b}) = (e^{ah}-1)^n e^{ax+b}$

If the interval of differencing is unity, prove that

$$\Delta \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}\left(\frac{1}{2x^2}\right)$$

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Sol: We have $\Delta f(x) = f(x+h) - f(x)$

Let $f(x) = \tan^{-1}\left(\frac{x-1}{x}\right)$

Given $h = 1$.

$$\Delta \tan^{-1}\left(\frac{x-1}{x}\right) = \Delta \tan^{-1}\left(1-\frac{1}{x}\right)$$

$$= \tan^{-1}\left(1-\frac{1}{x+1}\right) - \tan^{-1}\left(1-\frac{1}{x}\right)$$

$$= \tan^{-1}\left(\frac{\left(1-\frac{1}{x+1}\right)-\left(1-\frac{1}{x}\right)}{1+\left(1-\frac{1}{x+1}\right)\left(1-\frac{1}{x}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{x}-\frac{1}{x+1}}{1+\left(\frac{x}{x+1}\right)\left(\frac{x-1}{x}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x+1-x}{x(x+1)}}{\frac{(x+1)x+x(x-1)}{x(x+1)}}\right)$$

$$= \tan^{-1}\left(\frac{1}{2x^2}\right).$$

Evaluate $\Delta[f(x)g(x)]$.

Sol: We have $\Delta f(x) = f(x+h) - f(x)$.

$$\Delta[f(x)g(x)] = f(x+h)g(x+h) - f(x)g(x)$$

$$= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)$$

$$= f(x+h)\left[g(x+h)-g(x)\right] + g(x)\left[f(x+h)-f(x)\right]$$

$$= f(x+h)\Delta g(x) + g(x)\Delta f(x).$$

Evaluate $\Delta\left[\frac{f(x)}{g(x)}\right]$

Sol: We have $\Delta f(x) = f(x+h) - f(x)$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(x+h)\,g(x) - g(x+h)\,f(x)}{g(x+h)\,g(x)}$$

$$= \frac{f(x+h)\,g(x) - f(x)\,g(x) + g(x)\,f(x) - g(x+h)\,f(x)}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,[f(x+h) - f(x)] - f(x)\,[g(x+h) - g(x)]}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}.$$

Show that $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$.

Sol:- Let $y = f(x)$.

We know that the first forward difference is $\Delta y_k = y_{k+1} - y_k$.

Put $y_k = f(x_k) = f_k$, We get $\Delta f_k = f_{k+1} - f_k$.

The second difference is
$$\Delta^2 f_k = \Delta(\Delta f_k)$$
$$= \Delta(f_{k+1} - f_k)$$
$$= \Delta f_{k+1} - \Delta f_k.$$

$$\sum_{k=0}^{n-1} \Delta^2 f_k = \sum_{k=0}^{n-1} (\Delta f_{k+1} - \Delta f_k)$$
$$= (\Delta f_1 - \Delta f_0) + (\Delta f_2 - \Delta f_1) + \cdots + (\Delta f_n - \Delta f_{n-1})$$
$$= \Delta f_n - \Delta f_0.$$

If $f(x) = e^{ax}$ show that $\Delta^n f(x) = (e^{ah}-1)^n e^{ax}$. (11)

Sol: Given that $f(x) = e^{ax}$.

We have $\Delta f(x) = f(x+h) - f(x)$

$= e^{a(x+h)} - e^{ax}$. Here h is the step size.

We have to show that $\Delta^n f(x) = (e^{ah}-1)^n e^{ax}$.

This can be prooved by mathematical induction.

First we shall proove that this is true for $n=1$.

$(e^{ah}-1)^1 e^{ax} = e^{ah} \cdot e^{ax} - e^{ax}$.

$= e^{ah+ax} - e^{ax} = e^{a(x+h)} - e^{ax} = f(x+h) - f(x) = \Delta f(x)$.

$\therefore \Delta f(x_i) = f(x_i + h) - f(x_i)$.

Therefore, the result is true for $n=1$.

Assume that the problem is true for $n-1$.

Now Consider, $\Delta^n f(x) = \Delta^{n-1}[\Delta f(x)]$

$= \Delta^{n-1}[f(x+h) - f(x)]$

$= \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$

$= (e^{ah}-1)^{n-1} \cdot e^{a(x+h)} - (e^{ah}-1)^{n-1} e^{ax}$.

$= (e^{ah}-1)^{n-1}\left[e^{a(x+h)} - e^{ax}\right]$.

$= (e^{ah}-1)^{n-1}\left[e^{ax+ah} - e^{ax}\right]$

$= (e^{ah}-1)^{n-1}\left(e^{ax} e^{ah} - e^{ax}\right)$

$= (e^{ah}-1)^{n-1}(e^{ah}-1)\, e^{ax}$.

$= (e^{ah}-1)^n \cdot e^{ax}$.

$\therefore \Delta^n f(x) = (e^{ah}-1)^n e^{ax}$.

Show that $\Delta\left(\frac{f_i}{g_i}\right) = \frac{g_i \Delta f_i - f_i \Delta g_i}{g_i\, g_{i+1}}$ (12)

Sol:- We know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta\left(\frac{f_i}{g_i}\right) = \frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i}$$

$$\frac{g_i \Delta f_i - f_i \Delta g_i}{g_i\, g_{i+1}} = \frac{g_i (f_{i+1} - f_i) - f_i (g_{i+1} - g_i)}{g_i\, g_{i+1}}$$

$$= \frac{g_i f_{i+1} - f_i g_i - f_i g_{i+1} + f_i g_i}{g_i\, g_{i+1}} = \frac{g_i f_{i+1} - f_i g_{i+1}}{g_i\, g_{i+1}}$$

$$= \frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i}$$

$$\therefore \Delta\left(\frac{f_i}{g_i}\right) = \frac{g_i \Delta f_i - f_i \Delta g_i}{g_i\, g_{i+1}}$$

Show that $\Delta f_i^2 = (f_i + f_{i+1}) \Delta f_i$

Sol: We know that $\Delta f_k = f_{k+1} - f_k$

$$\Delta f_i^2 = f_{i+1}^2 - f_i^2$$

$$= (f_{i+1} - f_i)(f_{i+1} + f_i)$$

$$= (f_{i+1} + f_i) \Delta f_i$$

If the interval of differencing is unity prove that

$\Delta[x(x+1)(x+2)(x+3)] = 4(x+1)(x+2)(x+3)$

Sol:- Let $f(x) = x(x+1)(x+2)(x+3)$, We know that $\Delta f(x) = f(x+h) - f(x)$

$\Delta f(x) = f(x+1) - f(x)$ $[\because h = 1]$

$$= (x+1)(x+2)(x+3)(x+4) - x(x+1)(x+2)(x+3)$$

$$= 4(x+1)(x+2)(x+3)$$

Differences of a Polynomial :-

Result :- If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is a constant.

proof :- Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$

where $a_0, a_1, a_2, \dots a_n$ are constants and $a_0 \neq 0$.

If h is the step length, we know that $\Delta f(x) = f(x+h) - f(x)$.

$$\Delta f(x) = [a_0 (x+h)^n + a_1 (x+h)^{n-1} + \dots + a_{n-1}(x+h) + a_n]$$
$$- [a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n]$$

$$= a_0 [\{ {}^nC_0 x^n + {}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + \dots \} - x^n]$$
$$+ a_1 [\{ {}^{(n-1)}C_0 x^{n-1} + {}^{(n-1)}C_1 x^{n-2} h + {}^{(n-1)}C_2 x^{n-3} h^2 + \dots \} - x^{n-1}] + \dots + a_{n-1} h.$$

$$= a_0 [\{ x^n + n x^{n-1} h + \frac{n(n-1)}{2!} x^{n-2} h^2 + \dots \} - x^n]$$
$$+ a_1 [\{ x^{n-1} + (n-1) x^{n-2} h + \frac{(n-1)(n-2)}{2!} x^{n-3} h^2 + \dots \} - x^{n-1}] + \dots + a_{n-1} h.$$

$$= a_0 n h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \dots + b_{n-3} x + b_{n-2}$$

where $b_2, b_3, \dots b_{n-2}$ are constants. Here this polynomial is of degree $(n-1)$!

Thus, the first difference of a polynomial of nth degree is a polynomial of degree $(n-1)$.

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$
$$= \Delta [a_0 n h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \dots + b_{n-1} x + b_{n-2}]$$
$$= [a_0 n h (x+h)^{n-1} + b_2 (x+h)^{n-2} + b_3 (x+h)^{n-3} + \dots + b_{n-1}(x+h) + b_{n-2}]$$
$$- [a_0 n h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \dots + b_{n-1} x + b_{n-2}]$$
$$= a_0 n h [(x+h)^{n-1} - x^{n-1}] + b_2 [(x+h)^{n-2} - x^{n-2}] + \dots + b_{n-1}[(x+h) - x]$$
$$= a_0 n(n-1) h^2 x^{n-2} + c_3 x^{n-3} + \dots + c_{n-4} x + c_{n-3}$$

Where $c_3, c_4, \ldots c_{n-3}$ are constants. This polynomial is of degree $(n-2)$

Thus, the second difference of a polynomial of degree n is a polynomial of degree $(n-2)$. Continuing like this we get, $\Delta^n f(x)$.

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$\Delta^n f(x) = a_0\, n(n-1)(n-2) \ldots 2.1\, h^n = a_0 h^n (n!)$ Which is a constant.

Hence the result.

Note:- (i) As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0$;

$\Delta^{n+2} f(x) = 0, \ldots$

(ii) The converse of above result is also true. That is, if $\Delta^n f(x)$ is tabulated at equally spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n.

If the values of x are specified with step length h, evaluate

(i) $\Delta^3 [(1-x)(1-2x)(1+3x)]$ (ii) $\Delta^{10} [(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$.

Sol:- (i) Let $f(x) = (1-x)(1-2x)(1+3x)$

Which is a polynomial of degree 3 with 6 as the coefficient of x^3.

If $f(x)$ is a polynomial of degree n, a_0 is coefficient of x^n Then $\Delta^n f(x) = a_0 h^n n!$ Where h is the step length.

$\therefore \Delta^3 f(x) = \Delta^3 [(1-x)(1-2x)(1+3x)] = 6h^3 (3!) = 36 h^3$.

(ii) Let $f(x) = (1-x)(1-2x^2)(1-3x^3)(1-4x^4)$

Which is a polynomial of degree 10 with $(-1)(-2)(-3)(-4) = 24$ as the coefficient of x^{10}.

$\therefore \Delta^{10} f(x) = \Delta^{10} [(1-x)(1-2x^2)(1-3x^3)(1-4x^4)] = 24\, h^{10} (10!)$.

Find the second difference of the polynomial $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing $h = 2$.

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Sol: Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$.

We know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = f(x+2) - f(x) \quad \because h = 2$$

$$\Delta f(x) = [(x+2)^4 - 12(x+2)^3 + 42(x+2)^2 - 30(x+2) + 9] - [x^4 - 12x^3 + 42x^2 - 30x + 9]$$

$$= 8x^3 - 48x^2 + 56x + 28.$$

Second difference $\Delta^2 f(x) = \Delta[\Delta f(x)]$

$$= \Delta[8x^3 - 48x^2 + 56x + 28]$$

$$= [8(x+2)^3 - 48(x+2)^2 + 56(x+2) + 28] - [8x^3 - 48x^2 + 56x + 28]$$

$$= 48x^2 - 96x - 16.$$

If the interval of differencing is unity, prove that $\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x)f(x+1)}$

Sol: We know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = f(x+1) - f(x) \quad \because h = 1.$$

$$\Delta\left[\frac{1}{f(x)}\right] = \frac{1}{f(x+1)} - \frac{1}{f(x)}$$

$$= \frac{f(x) - f(x+1)}{f(x+1)f(x)}$$

$$= \frac{-[f(x+1) - f(x)]}{f(x+1)f(x)}$$

$$\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x+1)f(x)}$$

If the interval of differencing is unity, prove that $\Delta\left(\frac{2^x}{x!}\right) = \frac{2^x(1-x)}{(x+1)!}$

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Sol: Let $f(x) = \frac{2^x}{x!}$

We have $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = f(x+1) - f(x)$$

$$= \frac{2^{x+1}}{(x+1)!} - \frac{2^x}{x!}$$

$$= \frac{2^x \cdot 2}{(x+1)\,x!} - \frac{2^x}{x!} = \frac{2^x}{x!}\left(\frac{2}{x+1} - 1\right)$$

$$= \frac{2^x}{x!}\left(\frac{2-x-1}{(x+1)}\right)$$

$$= \frac{2^x(1-x)}{(x+1)!}$$

Prove that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Sol:- We know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log \frac{f(x+h)}{f(x)} = \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right].$$

Evaluate $\Delta(x + \cos x)$

Sol:- Here $f(x) = x + \cos x$

We have $\Delta f(x) = f(x+h) - f(x)$

$$\Delta f(x) = \Delta(x + \cos x) = \left[(x+h) + \cos(x+h)\right] - \left[x + \cos x\right]$$

$$= h + \cos(x+h) - \cos x.$$

Prove that $E\nabla = \Delta = \nabla E$ (17)

Sol:- We know that $\Delta f(x) = f(x+h) - f(x)$, $\nabla f(x) = f(x) - f(x-h)$

$$(E\nabla) f(x) = E(\nabla f(x)) = E[f(x) - f(x-h)]$$ $\because E f(x) = f(x+h)$

$$= E f(x) - E f(x-h)$$ $E^{-1} f(x) = f(x-h)$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

$$E\nabla = \Delta.$$

$$(\nabla E) f(x) = \nabla(E f(x)) = \nabla(f(x+h))$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

$$\nabla E = \Delta.$$

$$\therefore E\nabla = \Delta = \nabla E$$

Prove that $\delta E^{1/2} = \Delta$.

$$\delta u_{x+\frac{h}{2}} = (E^{1/2} - E^{-1/2}) u_{x+\frac{h}{2}}$$

$$= E^{1/2} u_{x+\frac{h}{2}} - E^{-1/2} u_{x+\frac{h}{2}}$$

$$= u_{x+h} - u_x$$

$$\delta u_{x+\frac{h}{2}} = \Delta u_x.$$

$$\delta E^{1/2} u_x = \Delta u_x$$

$$\delta E^{1/2} = \Delta.$$

Prove that $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$.

Sol: We know that $e^{hD} = E = 1+\Delta$.

Taking logarithm both sides, we get

$$\log_e e^{hD} = \log(1+\Delta)$$

$$hD \log_e e = \log(1+\Delta)$$

$$hD = \log(1+\Delta)$$

We have $\nabla = 1 - E^{-1} \Rightarrow E^{-1} = 1 - \nabla$.

$$e^{-hD} = 1 - \nabla \qquad \because e^{hD} = E$$

Taking logarithm bothsides, we get

$$\log_e e^{-hD} = \log_e (1-\nabla)$$

$$-hD \log_e e = \log_e (1-\nabla)$$

$$hD = -\log_e (1-\nabla)$$

$$\sinh hD = \frac{e^{hD} - e^{-hD}}{2} = \frac{E - E^{-1}}{2} = \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)\left(E^{1/2} - E^{-1/2}\right) = \mu\delta$$

$$\sinh hD = \mu\delta$$

$$hD = \sin^{-1}(\mu\delta).$$

Prove that $1 + \mu^2\delta^2 = \left(1 + \frac{1}{2}\delta^2\right)^2$

Sol: $$1 + \mu^2\delta^2 = 1 + \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)^2 \left(E^{1/2} - E^{-1/2}\right)^2$$

$$= 1 + \frac{\left[\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)\right]^2}{4}$$

$$= 1 + \frac{(E - E^{-1})^2}{4}$$

$$= \frac{4 + (E - E^{-1})^2}{4} = \left(\frac{E + E^{-1}}{2}\right)^2 \quad \text{——} \ (1)$$

$$\left(1 + \frac{1}{2}\delta^2\right)^2 = \left[1 + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2\right]^2 = \left[1 + \frac{1}{2}\left(E + E^{-1} - 2\right)\right]^2$$

$$= \left(\frac{E + E^{-1}}{2}\right)^2 \quad \text{——} \ (2)$$

From (1) and (2),

$$1 + \mu^2\delta^2 = \left(1 + \frac{1}{2}\delta^2\right)^2.$$

Prove that $E^{1/2} = \mu + \frac{1}{2}\delta$. (19) (36)

Sol: $\mu + \frac{1}{2}\delta = \frac{E^{1/2} + E^{-1/2}}{2} + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)$ $\quad \because \mu = \frac{E^{1/2} + E^{-1/2}}{2}$

$\delta = E^{1/2} - E^{-1/2}$

$\mu + \frac{1}{2}\delta = E^{1/2}$

Prove that $E^{-1/2} = \mu - \frac{1}{2}\delta$.

Sol: We know that $\mu = \frac{E^{1/2} + E^{-1/2}}{2}$, $\delta = E^{1/2} - E^{-1/2}$

$$\mu - \frac{1}{2}\delta = \frac{E^{1/2} + E^{-1/2}}{2} - \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)$$

$$= E^{-1/2}$$

Prove that $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$.

Sol: We have $\mu = \frac{E^{1/2} + E^{-1/2}}{2}$, $\delta = E^{1/2} - E^{-1/2}$, $\Delta = E - 1$.

$$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}\Delta(E^{-1} + 1) = \frac{1}{2}(E-1)(E^{-1}+1)$$

$$= \frac{1}{2}(E - E^{-1}) = \mu\delta.$$

Prove that $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$

Sol: We have $\Delta = E - 1$, $\delta = E^{1/2} - E^{-1/2}$

$$\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} = \frac{1}{2}\delta\left[\delta + 2\sqrt{1 + \frac{\delta^2}{4}}\right]$$

$$= \frac{1}{2}\delta\left[\delta + \sqrt{4 + \delta^2}\right]$$

$$= \frac{1}{2}\delta\left[\left(E^{1/2} - E^{-1/2}\right) + \sqrt{4 + \left(E^{1/2} - E^{-1/2}\right)^2}\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - E^{-1/2} + \sqrt{\left(E^{1/2} + E^{-1/2}\right)^2}\right]$$

$$= \frac{1}{2}\delta\left[E^{1/2} - E^{-1/2} + E^{1/2} + E^{-1/2}\right]$$

$$= \delta E^{1/2} = \left(E^{1/2} - E^{-1/2}\right)E^{1/2}$$

$$= E - 1$$

$$= \Delta.$$

Prove that $\nabla\Delta = \Delta - \nabla = \delta^2$.

sol:- $\Delta\nabla = (1-E^{-1})(E-1)$ $\qquad$ $\left[\because \begin{array}{l} \nabla = 1-E^{-1} \\ \Delta = E-1 \\ \delta = E^{1/2} - E^{-1/2} \end{array}\right.$

$$= E + E^{-1} - 2$$

$$= (E^{1/2} - E^{-1/2})^2$$

$$= \delta^2$$

$$\Delta - \nabla = (E-1) - (1-E^{-1})$$

$$= E + E^{-1} - 2$$

$$= (E^{1/2} - E^{-1/2})^2$$

$$= \delta^2$$

Prove that $(1+\Delta)(1-\nabla) = 1$.

sol: We have $\Delta = E-1$, $\nabla = 1 - E^{-1}$.

$$(1+\Delta)(1-\nabla) = E(1-\nabla)$$

$$= E(1-(1-E^{-1}))$$

$$= EE^{-1}$$

$$= 1.$$

Prove that $\mu\delta = \frac{1}{2}(\Delta+\nabla)$.

sol We have $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$, $\delta = E^{1/2} - E^{-1/2}$, $\Delta = E-1$, $\nabla = 1-E^{-1}$

$$\frac{1}{2}(\Delta+\nabla) = \frac{1}{2}(E-1+1-E^{-1})$$

$$= \frac{1}{2}(E-E^{-1})$$

$$= \frac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})$$

$$= \mu\delta.$$

Proove that (i) $y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n$ (ii) $y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$.

(21)

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Sol:- By definition $\nabla y_n = y_n - y_{n-1}$

$$\nabla y_{n-1} = y_{n-1} - y_{n-2}$$

$$\nabla^2 y_n = \nabla(\nabla y_n)$$

$$= \nabla(y_n - y_{n-1})$$

$$= \nabla y_n - \nabla y_{n-1}$$

$$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$$

$$\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$$

$$y_{n-2} = \nabla^2 y_n - y_n + 2y_{n-1}$$

$$= \nabla^2 y_n - y_n + y_{n-1} + y_{n-1} + y_n - y_n$$

$$= \nabla^2 y_n - (y_n - y_{n-1}) - (y_n - y_{n-1}) + y_n$$

$$= \nabla^2 y_n - \nabla y_n - \nabla y_n + y_n$$

$$y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n.$$

(ii) We have $\nabla y_{n-2} = y_{n-2} - y_{n-3}$ (OR) $y_{n-3} = y_{n-2} - \nabla y_{n-2}$

$$y_{n-3} = y_{n-2} - \nabla y_{n-2}$$

$$= (y_n - 2\nabla y_n + \nabla^2 y_n) - \nabla(y_n - 2\nabla y_n + \nabla^2 y_n)$$

$$= y_n - 2\nabla y_n + \nabla^2 y_n - \nabla y_n + 2\nabla^2 y_n - \nabla^3 y_n$$

$$= y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n.$$

If h is the step length, proove that $\Delta[f(x-h)\cdot\Delta g(x-h)] = \nabla[f(x)\cdot\Delta g(x)]$

Sol:- $\Delta[f(x-h)\cdot\Delta g(x-h)] = \Delta[f(x-h)\{g(x) - g(x-h)\}]$

$$= \Delta\{f(x-h)g(x)\} - \Delta\{f(x-h)g(x-h)\}$$

$$= f(x-h)\Delta g(x) + g(x+h)\Delta f(x-h)$$

$$- f(x-h)\Delta g(x-h) - g(x)\Delta f(x-h)$$

$$= f(x-h)\left[g(x+h)-g(x)\right] + g(x+h)\left[f(x)-f(x-h)\right]$$
$$- f(x-h)\left[g(x)-g(x-h)\right] - g(x)\left[f(x)-f(x-h)\right]$$

$$= f(x)\left[g(x+h)-g(x)\right] + f(x-h)\left[g(x-h)-g(x)\right]$$

$$= f(x)\,\Delta g(x) + f(x-h)\,\Delta g(x-h).$$

$$= \nabla\left[f(x).\,\Delta g(x)\right].$$

Since $\nabla\left[f(x)\,\Delta g(x)\right] = \nabla\left[f(x).\left(g(x+h)-g(x)\right)\right]$

$$= \nabla\left[f(x)\,g(x+h) - f(x)g(x)\right]$$

$$= \nabla\left(f(x)\,g(x+h)\right) - \nabla\left(f(x)\,g(x)\right)$$

$$= \left[f(x)\,g(x+h) - f(x-h)\,g(x)\right] - \left[f(x)g(x) - f(x-h).\,g(x-h)\right]$$

$$= f(x)\left[g(x+h)-g(x)\right] + f(x-h)\left[g(x-h)-g(x)\right]$$

$$\nabla\left[f(x).\,\Delta g(x)\right] = f(x)\,\Delta g(x) + f(x-h)\,\Delta g(x-h)$$

If the values of x are equally spaced, prove that

$(\Delta - \nabla) f(x) = \Delta\nabla f(x)$

Sol. We have $\Delta f(x) = f(x+h) - f(x)$, $\nabla f(x) = f(x) - f(x-h)$.

$$\Delta\nabla f(x) = \Delta\left[\nabla f(x)\right]$$
$$= \Delta\left[f(x) - f(x-h)\right]$$
$$= \Delta f(x) - \Delta f(x-h)$$
$$= \Delta f(x) - \left[f(x) - f(x-h)\right]$$
$$= \Delta f(x) - \nabla f(x)$$
$$= (\Delta - \nabla) f(x)$$

$$\Delta\nabla = \Delta - \nabla.$$

Proove that $\Delta^2 = E^2 - 2E + 1$.

Sol:- We know that $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$

$$\Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= [f(x+2h) - f(x+h)] - [f(x+h) - f(x)].$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

$$= E^2 f(x) - 2Ef(x) + f(x)$$

$$\Delta^2 f(x) = (E^2 - 2E + 1) f(x)$$

$$\Delta^2 = E^2 - 2E + 1.$$

Proove that $\Delta E = E\Delta$

Sol:- We know that $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$.

$$\Delta E f(x) = \Delta[Ef(x)]$$

$$= \Delta f(x+h)$$

$$= f(x+2h) - f(x+h)$$

$$E\Delta f(x) = E[\Delta f(x)]$$

$$= E[f(x+h) - f(x)]$$

$$= Ef(x+h) - Ef(x)$$

$$= f(x+2h) - f(x+h)$$

$$\therefore \Delta E f(x) = E\Delta f(x)$$

$$\Delta E = E\Delta.$$

Proove that $E^{-1}\Delta = \Delta E^{-1} = \nabla$.

Sol:- We have $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$, $\nabla f(x) = f(x) - f(x-h)$

$$\Delta E^{-1} f(x) = \Delta[E^{-1} f(x)] = \Delta f(x-h) = f(x) - f(x-h) = \nabla f(x)$$

$$\Delta E^{-1} f(x) = \nabla f(x)$$

$$\Delta E^{-1} = \nabla.$$

$$E^{-1}\Delta f(x) = E^{-1}[\Delta f(x)] = E^{-1}[f(x+h) - f(x)]$$
$$= E^{-1}f(x+h) - E^{-1}f(x)$$
$$= f(x) - f(x-h)$$
$$= \nabla f(x)$$
$$E^{-1}\Delta f(x) = \nabla f(x)$$
$$E^{-1}\Delta = \nabla.$$

Find (i) $\left(\frac{\Delta^2}{E}\right)f(x)$ (ii) $\frac{\Delta^2 f(x)}{Ef(x)}$ Deduce that $\left\{\left(\frac{\Delta^2}{E}\right)e^x\right\}\left\{\frac{Ee^x}{\Delta^2 e^x}\right\} = e^x$.

Sol: We know that $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$

$$\Delta = E - 1$$

$$\left(\frac{\Delta^2}{E}\right)f(x) = \frac{(E-1)^2}{E}f(x) = E^{-1}(E-1)^2 f(x) = E^{-1}(E^2+1-2E)f(x)$$
$$= (E + E^{-1} - 2)f(x)$$
$$= Ef(x) + E^{-1}f(x) - 2f(x)$$
$$= f(x+h) + f(x-h) - 2f(x)$$

Find $\frac{\Delta^2 f(x)}{Ef(x)}$

Sol: We know that $\Delta f(x) = f(x+h) - f(x)$, $E^n f(x) = f(x+nh)$

$$\frac{\Delta^2 f(x)}{Ef(x)} = \frac{(E-1)^2 f(x)}{Ef(x)} = \frac{(E^2-2E+1)f(x)}{Ef(x)} = \frac{f(x+2h) - 2f(x+h) + f(x)}{f(x+h)}$$

Now take $f(x) = e^x$ in the above results, we get.

$$\left(\frac{\Delta^2}{E}\right)e^x = e^{x+h} - 2e^x + e^{x-h}$$

$$\frac{\Delta^2 e^x}{Ee^x} = \frac{e^{x+2h} - 2e^{x+h} + e^x}{e^{x+h}}.$$

$$\left\{\left(\frac{\Delta^2}{E}\right)e^x\right\}\left\{\frac{Ee^x}{\Delta^2 e^x}\right\} = \frac{(e^{x+h} - 2e^x + e^{x-h})(e^{x+h})}{e^{x+2h} - 2e^{x+h} + e^x} = \frac{e^x(e^{x+2h} - 2e^{x+h} + e^x)}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= e^x.$$

Prove that $\Delta^3 y_2 = \nabla^3 y_5$ 25

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Sol: We have $\Delta = E-1, \quad \nabla = 1-E^{-1}$

$$\Delta^3 y_2 = (E-1)^3 y_2 = (E^3 - 3E^2 + 3E - 1) y_2$$

$$= E^3 y_2 - 3E^2 y_2 + 3E y_2 - y_2$$

$$= y_5 - 3y_4 + 3y_3 - y_2$$

$$\nabla^3 y_5 = (1-E^{-1})^3 y_5 = (1 - 3E^{-1} + 3E^{-2} - E^{-3}) y_5$$

$$= y_5 - 3E^{-1} y_5 + 3E^{-2} y_5 - E^{-3} y_5$$

$$= y_5 - 3y_4 + 3y_3 - y_2$$

$$\therefore \Delta^3 y_2 = \nabla^3 y_5 .$$

Prove that $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$

Sol:- We have $\Delta = E-1$.

$$(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2}) E^{1/2}$$

$$= E + 1$$

$$= E + 1 + 1 - 1$$

$$= 2 + E - 1$$

$$= 2 + \Delta .$$

If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -7845$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$ find y_4.

Sol:- Starting with y_1 instead of y_0, we note that $\Delta^6 y_1 = 0$.

We have $\Delta = E-1$.

$$\Delta^6 y_1 = 0 \Rightarrow (E-1)^6 y_1 = 0 \Rightarrow (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) y_1 = 0$$

$$E^6 y_1 - 6E^5 y_1 + 15E^4 y_1 - 20E^3 y_1 + 15E^2 y_1 - 6E y_1 + y_1 = 0.$$

$$y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0.$$

$$(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5) - 20y_4 = 0.$$

$$y_4 = \frac{1}{20}\left[(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5)\right]$$

$$= \frac{1}{20}\left[-784 - 6(686) + 15(1088)\right] = 571.$$

Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$ and $y_5 = 8$ find $\Delta^5 y_0$.

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sol: We know that $E = 1 + \Delta$

$$\Delta^5 y_0 = (E-1)^5 y_0 = (E^5 - {}^5C_1 E^4 + {}^5C_2 E^3 - {}^5C_3 E^2 + {}^5C_4 E - 1) y_0$$

$$= (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_0$$

$$= E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5E y_0 - y_0$$

$$= y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0$$

$$= 8 - 500 + 2000 - 810 + 60 - 3$$

$$= 755$$

If $y_0 = 5$, $y_1 = 11$, $y_2 = 22$, $y_3 = 40$, $y_5 = 140$ find y_4 given that the general term is represented by a fourth degree polynomial.

sol:- We know that $E = 1 + \Delta$.

Since y_n is represented by a 4^{th} degree polynomial, we have $\Delta^5 y_n = 0$.

i.e $(E-1)^5 y_n = 0$.

$$({}^5C_0 E^5 - {}^5C_1 E^4 + {}^5C_2 E^3 - {}^5C_3 E^2 + {}^5C_4 E - 1) y_n = 0.$$

$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_n = 0$$

$$E^5 y_n - 5E^4 y_n + 10E^3 y_n - 10E^2 y_n + 5E y_n - y_n = 0.$$

Take $n = 0$

$$E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5E y_0 - y_0 = 0.$$

$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0.$$

$$140 - 5y_4 + 400 - 220 + 55 - 5 = 0$$

$$5y_4 = 370$$

$$y_4 = 74.$$

Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

Sol:- $\left(\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}\right) f(x) = (\Delta\nabla^{-1} - \nabla\Delta^{-1}) f(x)$

$= \left(\Delta(1-E^{-1})^{-1} - \nabla(E-1)^{-1}\right) f(x)$

$= \left(\Delta\left(\frac{E-1}{E}\right)^{-1} - \nabla(E-1)^{-1}\right) f(x)$

$= \left(\Delta\left(\frac{E}{E-1}\right) - \frac{\nabla}{E-1}\right) f(x)$

$= \frac{1}{E-1}(\Delta E - \nabla) f(x)$

$= \frac{1}{E-1}\left((E-1)E - \left(1-\frac{1}{E}\right)\right) f(x)$

$= \frac{1}{E-1}\left((E-1)E - \left(\frac{E-1}{E}\right)\right) f(x)$

$= \left(E - \frac{1}{E}\right) f(x)$

$= (\Delta + \nabla) f(x)$ $\left[\because \Delta = E-1, \ \nabla = 1-E^{-1} \right.$ $\left. \Delta + \nabla = E - E^{-1}.\right]$

$\therefore \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$

Evaluate $(E^{-1}\Delta)x^3$ taking $h = 1$.

Sol:- Given that $f(x) = x^3$, $h = 1$

We know that $\Delta f(x) = f(x+h) - f(x)$

$\Delta f(x) = f(x+1) - f(x)$

$\Delta x^3 = (x+1)^3 - x^3$

$\Delta x^3 = x^3 + 1 + 3x^2 + 3x - \cancel{x^3}$

$\Delta x^3 = 3x^2 + 3x + 1$.

$E^{-1}(\Delta x^3) = E^{-1}(3x^2 + 3x + 1)$

We know that $E^{-1}f(x) = f(x-h)$.

$= 3(x-1)^2 + 3(x-1) + 1$

$= 3x^2 + 3x + 1$.

If $h=1$ is the step length, prove that

$$\Delta^n u_{x-n} = u_x - n u_{x-1} + \frac{n(n-1)}{2!} u_{x-2} + \cdots + (-1)^n u_{x-n}.$$

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Sol: Writing $u_x = f(x)$, we get

$$u_x - n u_{x-1} + \frac{n(n-1)}{2!} u_{x-2} + \cdots + (-1)^n u_{x-n}$$

$$= f(x) - n f(x-1) + \frac{n(n-1)}{2!} f(x-2) + \cdots + (-1)^n f(x-n).$$

We know that $E^{-n} f(x) = f(x-nh)$.

$$= f(x) - n E^{-1} f(x) + \frac{n(n-1)}{2!} E^{-2} f(x) + \cdots + (-1)^n E^{-n} f(x).$$

$$= \left\{1 - n E^{-1} + \frac{n(n-1)}{2!} E^{-2} + \cdots + (-1)^n E^{-n}\right\} f(x)$$

$$= (1 - E^{-1})^n f(x).$$

$$= \left(1 - \frac{1}{E}\right)^n f(x) = \left(\frac{E-1}{E}\right)^n f(x)$$

$$= \frac{\Delta^n}{E^n} f(x) = \Delta^n E^{-n} f(x)$$

$$= \Delta^n \left(E^{-n} f(x)\right)$$

$$= \Delta^n f(x-n) \qquad (\because E^{-n} f(x) = f(x-nh),\ h=1)$$

$$= \Delta^n u_{x-n}.$$

MODULE-V NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS AND NUMERICAL INTEGRATION

Differential Equation :- An equation involving differentials of one
dependent variable and its derivatives with respect to one or more
independent variables is called a differential equation.

Ordinary Differential Equation :- A differential equation is said
to be ordinary if the derivatives in the equation have reference
to only one a single independent variable.

Eg :- $\frac{dy}{dx} = x + y$, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2$.

Initial and Boundary value Problems :-

An ordinary differential equation of n^{th} order is of the form
$F(x, y, y', y'', y''', \ldots y^{(n)}) = 0$.
Its general solution will contain n arbitrary constants and it will be
of the form $f(x, y, c_1, c_2, c_3 \ldots c_n) = 0$.
To obtain its particular solution, n conditions must be given so that
the constants $c_1, c_2, c_3 \ldots c_n$ can be determined .
Problems in which $y, y', y'', \ldots y^{(n-1)}$ are all specified at the same value
of x say x_0 are called initial value problems .
If the conditions on y are prescribed at n distinct points then the
problems are called boundary value problems .
Problems in which function is prescribed at k different points and
$(n-k)$ derivatives are prescribed at the same point are called mixed
value problems .

Taylor series Method :-

Consider the initial value problem $y' = \frac{dy}{dx} = f(x,y)$, subject to $y = y_0$ when $x = x_0$.

$y(x)$ can be expanded about the point x_0 in a Taylor series in powers of $(x - x_0)$

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} \cdot y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} y^{(n)}(x_0) + \cdots$$

Where $y^i(x_0)$ is the ith derivative of $y(x)$ at $x = x_0$.

Let $x - x_0 = h$ (i.e $x = x_1 = x_0 + h$) we can write the Taylor's series as

$$y(x_1) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Substituting the values of $y_0, y_0', y_0'' \cdots$ etc in above equation we get the value of $y(x_1)$ (or) y_1.

Similarly expanding $y(x)$ in a Taylor series about the point x_1 we will get $y_2 = y_1 + \frac{h}{1!} y'(x_1) + \frac{h^2}{2!} y''(x_1) + \frac{h^3}{3!} y'''(x_1) + \cdots$

Similarly expanding $y(x)$ at a general point x_n, we will get

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

Working procedure :-

step(i) :- Compare the given diff. eqn. with $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

Identity $f(x,y)$, x_0 and y_0.

Identity the value of h (if not given)

step (ii) :-

Wkt Taylor series formula.

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{1!} y'_n + \frac{h^2}{2!} y''_n + \dots$$

Find y'', y''', y^{iv}, ...

step (iii) :- Find the values of y'', y''', y^{iv} at the point (x_0, y_0)

step (iv) :- sub. all these values in $y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \dots$

We get the value of y_1 at $x = x_1$.

Similarly we find y_2 at $x = x_2$, y_3 at $x = x_3$ ...

Find $y(0.1)$ using Taylor's series method given that $\frac{dy}{dx} = 1+xy$ and $y(0) = 1$.

sol:- Given that $\frac{dy}{dx} = 1+xy$ and $y(0) = 1$ —— ①

compare equation ① with $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

Here $f(x,y) = 1+xy$, $x_0 = 0$, $y_0 = 1$.

We find the value of y at $x = 0.1$.

The difference between $x = 0.1$ and $x_0 = 0$ is 0.1

so $h = 0.1$

We have. $x_1 = x_0 + h = 0.1$

We find the value of y at $x_1 = 0.1$ i.e y_1 (or) $y(0.1)$

WKT Taylor's series

$$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{1!} y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \cdots$$

$$y' = 1+xy$$

Diff. w.r.t 'x', we get

$$y'' = xy' + y.$$

$$y''' = xy'' + y' + y' = xy'' + 2y'$$

$$y^{iv} = xy''' + y'' + 2y'' = xy''' + 3y''$$

To find y_1 (or) $y(0.1)$:-

Put $n=0$ in ②, $y_1 = y(x_1) = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \frac{h^4}{4!} y^{iv}_0$ —— ③

At the pt $(x_0, y_0) = (0, 1)$

$$y'_0 = 1 + x_0 y_0 = 1.$$

$$y''_0 = x_0 y'_0 + y_0 = 1$$

$$y'''_0 = x_0 y''_0 + 2y'_0 = 0(1) + 2(1) = 2$$

$$y^{iv}_0 = x_0 y'''_0 + 3y''_0 = 0(2) + 3(1) = 3.$$

Substitute all these values in (1), we get.

$$y_1 = y(0.1) = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(3)$$

$$y_1 = y(0.1) = 1 + 0.1 + 0.005 + 0.000333 + 0.0000125$$

$$= 1.105$$

$$\therefore \quad y(0.1) = 1.105 .$$

Picards Method of Successive Approximations :-

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition

$y(x_0) = y_0$

Integrating the diff. eqn. from x_0 to x.

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$[y]_{x_0}^{x} = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\,dx \quad \text{--- (1)}$$

Here y is expressed under the integral sign, hence this is called an integral equation. This type of equations can be solved by the method of successive approximations.

The first approximation of y is obtained by putting $y = y_0$ in RHS of (1)

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

Similarly the second approximation of y is obtained by substituting $y = y_1$ in RHS of (1)

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

Proceeding in this way, we obtain $y_4, y_5, y_6, \dots y_n$.

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx.$$

Thus this method gives a sequence of approximations $y_1, y_2, y_3 \dots y_n$

The process of iteration is stopped when the values of y_{n-1} and y_n are approximately equal.

→ Given the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ with initial condition $y=0$ at $x=0$. Use Picard's method to obtain y at $x=0.25$, $x=0.5$ and $x=1$.

Sol: Given that $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ with Initial Condition $y=0$ at $x=0$ ——— (1)

Compare eqn (1) with $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

Here $f(x,y) = \frac{x^2}{y^2+1}$, $x_0 = 0$, $y_0 = 0$.

WKT Picard's formula. $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$.

1st Approximation :-

$n=1$, $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$.

$$y_1 = 0 + \int_0^x \frac{x^2}{y_0^2+1} dx$$

$$= \int_0^x x^2\, dx = \left[\frac{x^3}{3}\right]_0^x$$

$$y_1 = \frac{x^3}{3}$$

At $x = 0.25$ $\quad y_1 = \frac{(0.25)^3}{3} = 0.005208$

At $x = 0.5$ $\quad y_1 = \frac{(0.5)^2}{3} = 0.04166$

At $x = 1$, $\quad y_1 = \frac{1}{3} = 0.333$.

2nd Approximation :-

$n=2$, $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$

$$y_2 = 0 + \int_0^x \frac{x^2}{y_1^2+1} dx = \int_0^x \frac{x^2}{1+\left(\frac{x^3}{3}\right)^2} dx.$$

Put $\frac{x^3}{3} = t$

$\frac{3x^2}{3} dx = dt \Rightarrow x^2 dx = dt.$

$$y_2 = \int_0^{x^3/3} \frac{dt}{1+t^2} = \left[\tan^{-1}(t)\right]_0^{x^3/3}$$

$$y_2 = \tan^{-1}\left(\frac{x^3}{3}\right) - \tan^{-1}(0)$$

$$y_2 = \tan^{-1}\left(\frac{x^3}{3}\right)$$

At $x = 0.25$, $y_2 = \tan^{-1}\left(\frac{(0.25)^3}{3}\right) = \tan^{-1}(0.005208) = 0.005208$

At $x = 0.5$, $y_2 = \tan^{-1}\left(\frac{(0.5)^3}{3}\right) = \tan^{-1}(0.041667) = 0.04164$.

At $x = 1$, $y_2 = \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}(0.333) = 0.32145$

1st. and 2nd approximations are approximately equal.

$\therefore$ $y(0.25) = 0.005208$

$y(0.5) = 0.04164$

$y(1) = 0.32145$

Euler's Method :-

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition

$y(x_0) = y_0$.

$$dy = f(x,y)\,dx \quad \text{--- (1)}$$

Let us suppose that we want to find the approximate value of y say y_n at $x = x_n$ we divide the interval $[x_0, x_n]$ into n subintervals of equal length h say $x_0, x_1, x_2, \ldots x_n$.

Integrating (1) over $[x_0, x_1]$

$$\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx.$$

$$\left[y(x)\right]_{x_0}^{x_1} = \int_{x_0}^{x_1} f(x,y)\,dx$$

$$y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x,y)\,dx.$$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\,dx.$$

Let us put $f(x,y) = f(x_0,y_0)$ then

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0,y_0)\,dx.$$

$$y_1 = y_0 + f(x_0,y_0)\left[x\right]_{x_0}^{x_1}$$

$$y_1 = y_0 + f(x_0,y_0)(x_1 - x_0)$$

$$y_1 = y_0 + h f(x_0,y_0)$$

Similarly $y_2 = y_1 + h f(x_1,y_1)$

$$y_3 = y_2 + h f(x_2,y_2)$$

$$\cdots$$

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1}).$$

Working Procedure :-

Step (i) :- Compare given diff. eqn. with $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

Identify $f(x,y)$, x_0 and y_0 .

Identify the value of h (if not given)

Step (ii) WKT Eulers formula $y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$

$n=1$, $y_1 = y_0 + h f(x_0, y_0)$

Sub. the values of x_0, y_0 and h, we get y_1 (or) $y(x_1)$

$n=2$, $y_2 = y_1 + h f(x_1, y_1)$

Sub. the values of x_1, y_1 and h, we get y_2 (or) $y(x_2)$

Similarly we proceed in this we get $y_3, y_4 \cdots$

Solve by Euler's method, the equation $\frac{dy}{dx} = x+y$, $y(0)=0$ choose $h=0.2$, compute $y(0.4)$ and $y(0.6)$

Sol: Given that $\frac{dy}{dx} = x+y$, $y(0)=0$ —— ①

Compare eqn ① with $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

Here $f(x,y) = x+y$

$x_0 = 0$

$y_0 = 0$

Given that $h=0.2$

We have $x_1 = x_0 + h = 0+0.2 = 0.2$

$x_2 = x_1 + h = 0.2+0.2 = 0.4$

$x_3 = x_2 + h = 0.4+0.2 = 0.6$

→ We find the value of y at $x_1 = 0.2$ i.e y_1 or $y(0.2)$

→ We find the value of y at $x_2 = 0.4$ i.e y_2 or $y(0.4)$

→ We find the value of y at $x_3 = 0.6$ i.e y_3 or $y(0.6)$

→ We find the value of y

We know that Euler's formula.

$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})$$

To find y_1 (or) $y(0.2)$:-

$n=1$, $y_1 = y_0 + h\,f(x_0, y_0)$

$y_1 = 0 + (0.2)(x_0 + y_0)$

$y_1 = (0.2)(0+0) = 0$

$\therefore y_1 = y(0.2) = 0$.

To find y_2 (or) $y(0.4)$:-

$n=2$, $y_2 = y_1 + h\,f(x_1, y_1)$

$y_2 = 0 + (0.2)(x_1 + y_1)$

$y_2 = (0.2)(0.2+0)$

$y_2 = 0.04$

$\therefore \quad y_2 = y(0.4) = 0.04.$

To find y_3 (or) $y(0.6)$:-

$n = 3, \quad y_3 = y_2 + h \cdot f(x_2, y_2)$

$y_3 = (0.04) + (0.2)(x_2 + y_2)$

$y_3 = (0.04) + (0.2)(0.4 + 0.04)$

$y_3 = 0.128$

$\therefore \quad y_3 = y(0.6) = 0.128.$

Modified Euler's Method :-

Consider the differential eqn. $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$.

For find y_n at $x = x_n$ the modified Euler's formula is given by

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right]$$

To find $y(x_1) = y_1$ at $x = x_1 = x_0 + h$.

$$r = 1, \quad n = 1, 2, 3 \ldots$$

Using Euler's formula, $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$\ldots \ldots \ldots$$

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right]$$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another we will take the common values as y_1.

Now we have $\frac{dy}{dx} = f(x, y)$ with $y = y_1$ at $x = x_1$.

To get $y_2 = y(x_2) = y(x_1 + h)$ we use the above procedure again.

(1) Solve by Euler's modified method the equation $\frac{dy}{dx} = x+y$, $y(0)=0$ choose $h=0.2$ compute $y(0.4)$.

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Sol:- Given the differential equation is $\frac{dy}{dx} = x+y$

The initial condition is $y(0)=0$

$x_0 = 0,\ y_0 = 0$

Modified Euler's formula is

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right].$$

To find y_1 :-

$$x_1 = x_0 + h = 0+0.2 = 0.2$$

By Euler's formula, $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$$y_1^{(0)} = 0 + (0.2)(x_0+y_0) = (0.2)(0+0) = 0.$$

$$y_1^{(0)} = 0.$$

$y_1^{(1)}$ First Approximation :-

$$n=1,\quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$$

$$= y_0 + \frac{h}{2}\left[x_0+y_0 + x_1 + y_1^{(0)}\right]$$

$$= 0 + \frac{0.2}{2}\left[0+0+0.2+0\right] = 0.02$$

$$y_1^{(1)} = 0.02$$

Second Approximation :-

$$n=2,\quad y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$= y_0 + \frac{h}{2}\left[x_0+y_0 + x_1 + y_1^{(1)}\right]$$

$$= 0 + \frac{0.2}{2}\left[0+0+0.2+0.02\right] = 0.1\left[0.22\right] = 0.022$$

$$y_1^{(2)} = 0.022$$

Third Approximation :-

$$n=3\quad y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 0 + \frac{0.2}{2}\left[x_0+y_0 + x_1 + y_1^{(2)}\right] = 0.1\left[0+0+0.2+0.022\right]$$

$$y_1^{(3)} = 0.1(0.222) = 0.0222.$$

At $x = 0.2$, second and third Approximations are approximately equal.

$\therefore y(0.2) = 0.0222$

To find y_2 :-

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

By Euler's formula. $y_2^{(0)} = y_1 + h f(x_1, y_1)$

$y_2^{(0)} = 0.0222 + (0.2)(0.2 + 0.0222)$

$y_2^{(0)} = 0.06664$.

First Approximation :-

$r = 2, \quad n = 1, \qquad y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$

~~= 0.0222 + 0.2~~

$= y_1 + \frac{h}{2}\left[x_1 + y_1 + x_2 + y_2^{(0)}\right]$

$= 0.0222 + \frac{0.2}{2}\left[0.2 + 0.0222 + 0.4 + 0.06664\right]$

$= 0.0222 + 0.1\left[0.68884\right]$

$y_2^{(1)} = 0.091084$.

Second Approximation :-

$n = 2 \qquad y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$

$= y_1 + \frac{h}{2}\left[x_1 + y_1 + x_2 + y_2^{(1)}\right]$.

$= 0.0222 + \frac{0.2}{2}\left[0.2 + 0.0222 + 0.4 + 0.091084\right]$

$= 0.0222 + 0.1\left[0.713284\right.$

$y_2^{(2)} = 0.0935284$.

Third Approximation :-

$n = 3 \quad y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$

$= y_1 + \frac{h}{2}\left[x_1 + y_1 + x_2 + y_2^{(2)}\right]$

$= 0.0222 + \frac{0.2}{2}\left[0.2 + 0.0222 + 0.4 + 0.0935284\right]$

$= 0.0222 + 0.1\left[0.7157284\right] = 0.09377284$

At $x = 0.4$, Second and third approximations are approximately equal. $\therefore y(0.4) = 0.0938$

(8) Given $\frac{dy}{dx} - \sqrt{xy} = 2$ and $y(1) = 1$, Find the value of $y(1.5)$ in steps of 0.25 using Euler's modified method. 183 (15)

Sol: G.T the differential eqn is $\frac{dy}{dx} - \sqrt{xy} = 2$

$$\frac{dy}{dx} = 2 + \sqrt{xy}$$

The Initial condition is $y(1) = 1$

$$x_0 = 1 \quad y_0 = 1.$$

Here $f(x,y) = 2 + \sqrt{xy}$ $\quad h = 0.25$.

Modified Euler's formula

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right].$$

To find y_1 :-

$$x_1 = x_0 + h = 1 + 0.25 = 1.25.$$

By Euler's formula, $y_1^{(0)} = y_0 + hf(x_0, y_0)$

$$y^{(0)} = y_0 + h(2 + \sqrt{x_0 y_0})$$

$$y_1^{(0)} = 1 + 0.25(2 + \sqrt{1}) = 1 + 0.25(3) = 1.75$$

First Approximation :-

$n=1$ $\quad y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$$= y_0 + \frac{h}{2}\left[2 + \sqrt{x_0 y_0} + 2 + \sqrt{x_1 y_1^{(0)}}\right]$$

$$= 1 + \frac{0.25}{2}\left[2 + \sqrt{(1)(1)} + 2 + \sqrt{(1.75)(1.25)}\right]$$

$$= 1 + \frac{0.25}{2}\left[5 + \sqrt{(1.75)(1.25)}\right]$$

$$y_1^{(1)} = 1.8099.$$

Second Approximation :-

$n=2$, $\quad y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[2 + \sqrt{x_0 y_0} + 2 + \sqrt{x_1 y_1^{(1)}}\right]$$

$$= 1 + \frac{0.25}{2}\left[4 + \sqrt{(1)(1)} + \sqrt{(1.25)(1.8099)}\right]$$

$$= 1 + \frac{0.25}{2}\left[5 + \sqrt{(1.25)(1.8099)}\right]$$

$$y_1^{(2)} = 1.8130.$$

Third Approximation :-

$n=3 \quad y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$

$= y_0 + \frac{h}{2}\left[2+\sqrt{x_0 y_0} + 2+\sqrt{x_1 y_1^{(2)}}\right]$

$= 1 + \frac{0.25}{2}\left[4+\sqrt{(1)(1)} + \sqrt{(1.25)(1.8130)}\right]$

$= 1 + \frac{0.25}{2}\left[5+\sqrt{(1.25)(1.8130)}\right]$

$y_1^{(3)} = 1.8132$

At $x = 1.25$, second and third approximations are approximately equal.

$\therefore \ y(1.25) = 1.8132$.

To find y_2 :-

$x_2 = x_1 + h = 1.25 + 0.25 = 1.5$.

$y_2^{(0)} = y_1 + hf(x_1, y_1)$

$= y_1 + h(2+\sqrt{x_1 y_1})$

$= 1.8132 + (0.25)(2+\sqrt{(1.25)(1.8132)})$

$= 1.8132 + (0.25)(2+\sqrt{(1.25)(1.8132)})$

$y_2^{(0)} = 2.6896$.

First Approximation :-

$n=1, \quad y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$

$= y_1 + \frac{h}{2}\left[2+\sqrt{x_1 y_1} + 2+\sqrt{x_2 y_2^{(0)}}\right]$

$= 1.8132 + \frac{0.25}{2}\left[4+\sqrt{(1.25)(1.8132)} + \sqrt{(1.5)(2.6896)}\right]$

$y_2^{(1)} = 2.7525$

Second Approximation :-

$n=2, \quad y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$

$= y_1 + \frac{h}{2}\left[2+\sqrt{x_1 y_1} + 2 + \sqrt{x_2 y_2^{(1)}}\right]$

$= 1.8132 + \frac{0.25}{2}\left[4+\sqrt{(1.25)(1.8132)} + \sqrt{(1.5)(2.7525)}\right]$

$y_2^{(2)} = 2.7554$

Third Approximation :-

$n=3$ $\quad y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$

$$= y_1 + \frac{h}{2}\left[2 + \sqrt{x_1 y_1} + 2 + \sqrt{x_2 y_2^{(2)}}\right]$$

$$= 1.8132 + \frac{0.25}{2}\left[2 + \sqrt{(1.25)(1.8132)} + 2 + \sqrt{(1.5)(2.7554)}\right]$$

$$= 1.8132 + \frac{0.25}{2}\left[4 + \sqrt{(1.25)(1.8132)} + \sqrt{(1.5)(2.7554)}\right]$$

$$y_2^{(3)} = 2.7555$$

At $x = 1.5$, second and third approximations are approximately equal.

$\therefore y(1.5) = 2.7555$

Runge Kutta Methods :-

First order Runge kutta Method :-

We know that, By Euler's method.

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h y_0' \quad (\because y' = f(x,y))$$

By Taylor's series

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \dots$$

Euler's method is same as the Taylor's series solution upto the term in h.

$\Rightarrow$ Euler's method is Runge Kutta method of first order.

Second order Runge Kutta Method :-

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

$$\frac{dy}{dx} = f(x,y)$$
$$dy = f(x,y)\,dx.$$

Integrating the above equation over $[x_0, x_1]$ using Trapezoidal rule.

$$\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

$$[y]_{x_0}^{x_1} = \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right]$$

$$y(x_1) - y(x_0) = \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1)\right].$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_0+h,\ y_0 + h f(x_0,y_0))\right]$$

$$y_1 = y_0 + \frac{1}{2}\left[h f(x_0,y_0) + h f(x_0+h,\ y_0 + h f(x_0,y_0))\right] \quad \text{--- (1)}$$

$x_1 = x_0 + h \qquad y_1 = y_0 + h f(x_0,y_0).$

Now we put $k_1 = h f(x_0,y_0).$

$$k_2 = h f(x_0+h,\ y_0 + h f(x_0,y_0))$$
$$k_2 = h f(x_0+h,\ y_0 + k_1).$$

Then from (1).

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

similarly for finding y_2, $y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h f(x_1, y_1)$

$k_2 = h f(x_1 + h, y_1 + k_1)$

In General. $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + h, y_n + k_1)$ $n = 0, 1, 2, \ldots$

which is called second order Runge kutta formula.

(II) Fourth order

Third order Runge kutta Method :—

The third order Runge kutta method formula is.

$$y_{n+1} = y_n + \frac{(k_1 + 4k_2 + k_3)}{6} \quad n = 0, 1, 2, \ldots$$

where $k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$.

$k_3 = h f(x_n + h, y_n + 2k_2 - k_1)$.

Fourth order Runge kutta Method :—

The fourth order Runge kutta method formula is

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$n = 0, 1, 2, \ldots$

where $k_1 = h f(x_n, y_n)$.

$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = h f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = h f(x_n + h, y_n + k_3)$.

(1) Given $\frac{dy}{dx} = -y$, $y(0)=1$, using Runge Kutta method of second order find the values of y at $x=0.1$ at $x=0.2$.

(13)

Sol: Given the differential equation is $\frac{dy}{dx} = -y$.

Here $f(x,y) = -y$

The initial condition is $y(0) = 1$

$x_0 = 0 \quad y_0 = 1.$

The second order Runge Kutta formula is.

$$y_{n+1} = y_n + \frac{1}{2}(k_1+k_2)$$

$$k_1 = h\, f(x_n, y_n)$$

$$k_2 = h\, f(x_n+h, y_n+k_1)$$

<u>To find y_1 :-</u>

$n=0, \quad y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = hf(x_0+h, y_0+k_1)$$

Here $h = 0.1,$

$$k_1 = h(-y_0)$$

$$k_1 = (0.1)(-1) = -0.1$$

$$k_2 = h(-(y_0+k_1)) = h(-y_0-k_1)$$

$$k_2 = 0.1(-1+0.1) = (0.1)(-0.9)$$

$$k_2 = -0.09.$$

$$y_1 = y(0.1) = y_0 + \frac{1}{2}(k_1+k_2)$$

$$= 1 + \frac{1}{2}(-0.1-0.09)$$

$$= 1+\frac{1}{2}(-0.19)$$

$$y_1 = 0.905$$

<u>To find y_2 :-</u>

$n=1, \quad y_2 = y_1 + \frac{1}{2}(k_1+k_2)$

$$k_1 = h f(x_1, y_1).$$

$$k_2 = h\, f(x_1+h, y_1+k_1).$$

$$k_1 = h(-y_1)$$

$$k_1 = (0.1)(-0.905) = -0.0905$$

$k_2 = h \cdot f(x_1+h, y_1+k_1)$

$= h(-y_1 - k_1)$

$= (0.1)(-0.905 + 0.0905)$

$k_2 = -0.08145$

$y_2 = y(0.2) = y_1 + \frac{1}{2}(k_1+k_2)$

$y_2 = 0.905 + \frac{1}{2}(-0.0905 - 0.08145)$

$y_2 = 0.819025$

(2) Using Runge Kutta method of second order, compute $y(2.5)$ from

$\frac{dy}{dx} = \frac{x+y}{x}$ $\quad y(2) = 2$ taking $h = 0.25$.

Sol:- G/T the differential equation is $\frac{dy}{dx} = \frac{x+y}{x}$

Here $f(x,y) = \frac{x+y}{x}$

The initial condition is $y(2) = 2$ i.e. $x_0 = 2$, $y_0 = 2$.

$h = 0.25$

The second order Euler's formula is.

$y_{n+1} = y_n + \frac{1}{2}(k_1+k_2)$

$k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n+h, y_n+k_1)$

To find y_1 :-

$n=0$, $\quad y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$k_1 = h f(x_0, y_0)$

$k_2 = h f(x_0+h, y_0+k_1)$

$k_1 = h f(x_0, y_0)$

$k_1 = h \cdot \left(\frac{x_0+y_0}{x_0}\right)$

$k_1 = (0.25)\left(\frac{2+2}{2}\right) = (0.25)2 = 0.5$

$k_2 = h f(x_0+h, y_0+k_1)$

$k_2 = h f(2.25, 2.5) = (0.25)\left(\frac{2.25+2.5}{2.25}\right)$

$k_2 = 0.528$

$$y_1 = y(2.25) = y_0 + \frac{1}{2}(k_1+k_2)$$

$$y_1 = 2 + \frac{1}{2}(0.5+0.528)$$

$$y_1 = 2.514.$$

To find y_2 :-

$n=2$,
$$y_2 = y_n + \frac{1}{2}(k_1+k_2)$$
$$k_1 = h f(x_1, y_1)$$
$$k_2 = h f(x_1+h, y_1+k_1)$$

$$k_1 = h f(x_1, y_1)$$
$$k_1 = h\left(\frac{x_1+y_1}{x_1}\right)$$
$$= (0.25)\left(\frac{2.25+2.514}{2.25}\right)$$
$$k_1 = 0.5293$$

$$k_2 = h f(x_1+h, y_1+k_1)$$
$$k_2 = h f(2.25+0.25, 2.514+0.5293)$$
$$k_2 = h f(2.5, 3.0433)$$
$$k_2 = (0.25)\left(\frac{2.5+3.0433}{2.5}\right)$$
$$k_2 = 0.55433.$$

$$y_2 = \frac{1}{2}(k_1+k_2) + y_1$$
$$y_2 = \frac{1}{2}(0.5293+0.55433) + 2.514$$
$$y_2 = 3.055815.$$

Obtain the values of y at $x=0.1$ and 0.2 using third order Runge Kutta method, given $\frac{dy}{dx} = -y$, $y(0)=1$.

Sol:- GIT the differential eqn is $\frac{dy}{dx} = -y$

Here $f(x,y) = -y$.

The initial condition is $y(0)=1$.

Third order Runge kutta formula is

$$y_{n+1} = y_n + \frac{(k_1+4k_2+k_3)}{6} \quad n=0,1,2,\ldots$$

where
$$k_1 = h f(x_n, y_n)$$
$$k_2 = h f\left(x_n+\frac{h}{2}, y_n+\frac{k_1}{2}\right)$$
$$k_3 = h f(x_n+h, y_n+2k_2-k_1).$$

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To find y_1 :-

$$n=0, \quad y_1 = y_0 + \left(\frac{k_1+4k_2+k_3}{6}\right)$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = h\,f(x_0+\tfrac{h}{2},\ y_0+\tfrac{k_1}{2})$$

$$k_3 = h\,f(x_0+h,\ y_0+2k_2-k_1)$$

$$k_1 = h\,f(x_0,y_0)$$

$$k_1 = h(-y_0) = (0.1)(-1) = -0.1$$

$$k_2 = h\,f(x_0+\tfrac{h}{2},\ y_0+\tfrac{k_1}{2})$$

$$k_2 = h\left(-(y_0+\tfrac{k_1}{2})\right) = h\left(-y_0-\tfrac{k_1}{2}\right)$$

$$k_2 = (0.1)\left(-1+\tfrac{0.1}{2}\right)$$

$$k_2 = -0.095.$$

$$k_3 = h\,f(x_0+h,\ y_0+2k_2-k_1)$$

$$k_3 = h\left(-(y_0+2k_2-k_1)\right)$$

$$k_3 = (0.1)\left(-(1-2(0.095)+0.1)\right)$$

$$k_3 = (0.1)(-0.91)$$

$$k_3 = -0.091.$$

$$y_1 = y_0 + \left(\frac{k_1+4k_2+k_3}{6.}\right)$$

$$= 1 + \frac{(-0.1+4(-0.095)-0.091)}{6}$$

$$y_1 = 0.905$$

To find y_2 :-

$$x_1 = x_0+h = 0+0.1 = 0.1 \quad y_1 = 0.905$$

$$n=1, \quad y_2 = y_1 + \left(\frac{k_1+4k_2+k_3}{6}\right)$$

Where $k_1 = h\,f(x_1, y_1)$

$$k_2 = h\,f(x_1+\tfrac{h}{2},\ y_1+\tfrac{k_1}{2})$$

$$k_3 = h\,f(x_1+h,\ y_1+2k_2-k_1)$$

$k_1 = h f(x_1, y_1)$

$k_1 = h(-y_1) = (0.1)(-0.905)$

$k_1 = -0.0905$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= h(-(y_1 + \frac{k_1}{2}))$

$= (0.1)(-(0.905 + \frac{(-0.0905)}{2}))$

$k_2 = -0.085975$

$k_3 = h f(x_n + h, y_n + 2k_2 - k_1)$

$k_3 = h f(x_1 + h, y_1 + 2k_2 - k_1)$

$k_3 = h(-(y_1 + 2k_2 - k_1))$

$k_3 = (0.1)(-(0.905 + 2(-0.085975) + 0.0905))$

$k_3 = -0.082355$

$y_2 = y(0.2) = y_1 + \frac{1}{6}(k_1 + 4k_2 + k_3)$

$= 0.905 + \frac{1}{6}(-0.0905 + 4(-0.085975) + 0.082355)$

$= 0.905 + \frac{1}{6}($

$y_2 = 0.818874.$

14) Given that $y' = y - x$, $y(0) = 2$ find $y(0.2)$ using Runge Kutta method Take $h = 0.1$.

Sol:- G.T the differential equation $y' = y - x$.

Here $f(x, y) = y - x$.

The initial condition is $y(0) = 2$

$x_0 = 0, \quad y_0 = 2, \quad h = 0.1$

Fourth order Runge Kutta formula is

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = h f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = h f(x_n + h, y_n + k_3)$

To find y_1 :-

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$n=0, \quad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$\rightarrow k_1 = hf(x_0, y_0)$

$k_1 = h(y_0 - x_0)$

$k_1 = (0.1)(2-0) = 0.2$

$\rightarrow k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= hf(0 + \frac{0.1}{2}, 2 + \frac{0.2}{2})$

$= hf(0.05, 2.1)$

$= (0.1)(2.1 - 0.05)$

$k_2 = 0.205$

$\rightarrow k_3 = hf(x_0 + \frac{h}{2}; y_0 + \frac{k_2}{2})$

$= hf(0 + \frac{0.1}{2}, 2 + \frac{0.205}{2})$

$k_3 = hf(0.05, 2.10025)$

$k_3 = (0.1)(2.0525 - 0.05)$

$k_3 = 0.20525$

$\rightarrow k_4 = hf(x_0 + h, y_0 + k_3)$

$k_4 = hf(0.1, 2 + 0.20525)$

$k_4 = 0.210525$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$

$= 2 + \frac{1}{6}(0.2 + (0.205 + 0.20525)2 + 0.210525)$

$= 2 + \frac{1}{6}($

$y_1 = 2.2052$

To find y_2 :-

$n=1, \quad y_2 = y_1 + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$

$k_1 = h f(x_1, y_1)$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$k_4 = h f(x_1 + h, y_1 + k_3)$

$h = 0.1 \quad x_1 = 0.1 \quad y_1 = 2.2052$

$k_1 = hf(x_1, y_1)$

$k_1 = h(y_1 - x_1) = (0.1)(2.2052 - 0.1) = (0.1)(2.1052) = 0.21052$

$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= hf(0.1 + \frac{0.1}{2}, 2.2052 + \frac{0.21052}{2})$

$= hf(0.15, 2.2052 + 0.10526) = hf(0.15, 2.31046)$

$= (0.1)(2.31046 - 0.15) = (0.1)(2.16046)$

$k_2 = 0.216046$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$= hf(0.1 + \frac{0.1}{2}, 2.2052 + \frac{0.216046}{2})$

$= hf(0.1 + 0.05, 2.2052 + 0.108023)$

$= hf(0.15, 2.313223)$

$k_3 = (0.1)(2.313223 - 0.15) = (0.1)(2.163223)$

$k_3 = 0.2163223$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= hf(0.1 + 0.1, 2.2052 + 0.2163223)$

$= hf(0.2, 2.4215223)$

$= (0.1)(2.4215223 - 0.2)$

$k_4 = 0.22215223$

$y_2 = y_1 + \left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right)$

$y_2 = 2.2052 + \frac{(0.21052 + 2(0.216046 + 0.2163223) + 0.22215223)}{6}$

$y_2 = 2.421496483$

$y_2 = 2.4215$

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Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0)=1$, compute $y(0.2)$ by using fourth order Runge Kutta method by taking $h=0.2$.

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Sol: G/T $\frac{dy}{dx} = \frac{y-x}{y+x}$

Here $y(0)=1$

$x_0 = 0,\ y_0 = 1 \quad h = 0.2$

Here $f(x,y) = \frac{y-x}{y+x}$

The fourth order Runge Kutta formula is

$$y_{n+1} = y_n + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \tfrac{h}{2}, y_n + \tfrac{k_1}{2})$$

$$k_3 = hf(x_n + \tfrac{h}{2}, y_n + \tfrac{k_2}{2})$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

To find y_1 :-

$n=0$, $y_1 = y_0 + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2})$$

$$k_3 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2})$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k_1 = hf(x_0, y_0)$$

$$k_1 = h\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$k_1 = 0.2\left(\frac{1-0}{1+0}\right) = 0.2$$

$$k_2 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2})$$

$$k_2 = hf\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right) = hf(0.1, 1.1)$$

$$k_2 = (0.2)\left(\frac{1.1 - 0.1}{1.1 + 0.1}\right) = (0.2)\left(\frac{1}{1.2}\right) = 0.16667$$

$$k_3 = hf(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_2}{2})$$

$$= hf\left(0 + \frac{0.2}{2}, 1 + \frac{0.16667}{2}\right) = hf(0.1, 1.083335)$$

$$k_3 = (0.2)\left(\frac{1.083335 - 0.1}{1.083335 + 0.1}\right) = (0.2)\left(\frac{0.983335}{1.183335}\right) = 0.166197$$

$k_4 = h.f(x_0+h, y_0+k_3)$

$= hf(0+0.2, 1+0.166197)$

$= h.f(0.2, 1.166197)$

$k_4 = (0.2)\left(\frac{1.166197-0.2}{1.166197+0.2}\right) = (0.2)\left(\frac{0.966197}{1.366197}\right)$

$k_4 = 0.14144.$

$y_1 = y_0 + \frac{(k_1+2k_2+2k_3+k_4)}{6}$

$= 1 + \frac{0.2 + 2(0.16667 + 0.166197) + 0.14144)}{6}$

$= 1 + \cancel{0.503587}\ 0.16786$

$y_1 = \cancel{1.503587}\ 1.16786.$

Numerical Integration :—

The method finding the value of an integral of the form $\int_a^b f(x)dx$ using numerical techniques is called Numerical integration.

Let $y = f(x)$ be a function on $[a,b]$. If the function $f(x)$ is defined explicitly and the integral of $f(x)$ can be calculated by the usual methods then the definite integral $\int_a^b f(x)\,dx$ can be found easily.

If the function is given in tabular form and the integral of $f(x)$ is difficult to find then numerical integration is needed.

Numerical integration is used to obtain approximate answers for definite integrals that can not be solved analytically. It is a process of finding the numerical value of a definite integral $I = \int_a^b f(x)dx$ when the function $y = f(x)$ is not known explicitly.

Importance of Numerical Methods :—

The Numerical methods are important because finding an analytical procedure to solve an equation may ~~be~~ not be always available. In such cases Numerical Analysis provides approximate solutions.

(i) To solve the ordinary differential equations.

(ii) To solve the Algebraic and Transcendental Equations.

(iii) To solve Numerical Integration and Differentiation.

Trapezoidal Rule :-

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left(y_0 + y_1 + y_2 + y_3 + y_4 + \cdots + y_{n-1} + y_n\right)$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]$$

where $h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$

(1) Calculate the value $\int_0^1 \frac{x}{1+x}\,dx$ correct to 3 significant figures taking 6 intervals by trapezoidal rule.

sol:- Let $y = \frac{x}{1+x}$, $x_0 = 0$ $x_n = 1$ $n = 6$

$$h = \frac{b-a}{n} = \frac{x_n - x_0}{n} = \frac{1-0}{6} = \frac{1}{6}.$$

x	$y = \frac{x}{1+x}$
$x_0 = 0$	$y_0 = \frac{x_0}{1+x_0} = \frac{0}{1+0} = 0$
$x_1 = \frac{1}{6}$	$y_1 = \frac{x_1}{1+x_1} = \frac{1/6}{1+1/6} = \frac{1}{7} = 0.1428$
$x_2 = \frac{1}{3}$	$y_2 = \frac{x_2}{1+x_2} = \frac{1/3}{1+\frac{1}{3}} = \frac{1}{4} = 0.25$
$x_3 = \frac{1}{2}$	$y_3 = \frac{x_3}{1+x_3} = \frac{1/2}{1+1/2} = \frac{1}{3} = 0.333$
$x_4 = \frac{2}{3}$	$y_4 = \frac{x_4}{1+x_4} = \frac{2/3}{1+2/3} = \frac{2}{5} = 0.4$
$x_5 = \frac{5}{6}$	$y_5 = \frac{x_5}{1+x_5} = \frac{5/6}{1+5/6} = \frac{5}{11} = 0.4545$
$x_6 = 1$	$y_6 = \frac{x_6}{1+x_6} = \frac{1}{1+1} = \frac{1}{2} = 0.5$

Trapezoidal Rule.

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$\int_0^1 \frac{x}{1+x}\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+\cdots+y_5)\right]$$

$$= \frac{1}{12}\left[(0+0.5)+2(0.1428+0.25+0.333+0.4+0.4545)\right]$$

$$\int_0^1 \frac{x}{1+x}\,dx \approx 0.30505$$

Simpson's $\frac{1}{3}$ - Rule :—

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+\cdots+y_{n-2})\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(\text{Sum of odd ordinates})+2(\text{Sum of even ordinates})\right].$$

Note:- It can be applied only when the given interval $[a,b]$ is subdivided into even no. of sub intervals each of width h and with in any two consecutive subintervals the interpolating polynomial $\phi(x)$ is of degree 2.

(1) The velocity of a train which starts from rest is given by the following table.

t min	2	4	6	8	10	12	14	16	18	20
V km/hr	16	28.8	40	46.4	51.2	32	17.6	8	3.2	0

Estimate total distance run in 20 min.

Sol:- We know that

The rate of change of displacement is called velocity.

i.e $\frac{ds}{dt} = v$

$$ds = v\,dt$$

$$s = \int_0^{20} v\,dt$$

Since the train starts from rest $\Rightarrow$ $v=0$ at $t=0$

$t_0 = 0$, $v_0 = 0$.

$t_0 = 0$ $\quad t_n = 20$.

$$h = \frac{20-0}{10} = 2 \text{ min} = \frac{2}{60} = \frac{1}{30} \text{ hrs}.$$

Simpson's $\frac{1}{3}$ Rule.

$$s = \int_0^{20} v\,dt = \frac{h}{3}\left[(y_0+y_{10}) + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$$

$$= \frac{1}{90}\left[(0+0) + 4(16+40+51.2+17.6+3.2) + 2(28.8+46.4+32+8)\right]$$

$$= \frac{1}{90}\left[4(128) + 2(115.2)\right]$$

$$= 8.25 \text{ km}$$

The distance run by train in 20 min $= 8.25$ km.

Simpson's $\frac{3}{8}$ - Rule :—

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+\cdots+y_{n-1}) + 2(y_3+y_6+y_9+\cdots+y_{n-3})\right]$$

Note:- It can be applied when the range $[a,b]$ is divided into no. of sub intervals, which is a multiple of 3.

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(1) Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using Simpson's $\frac{1}{3}$ and Simpson's $\frac{3}{8}$ Rule, and Trapezoidal Rule. Also compare it with its exact value. Hence obtain the approximate value of π in each.

sol: Let $y = f(x) = \frac{1}{1+x^2}$.

Given $x_0 = 0$ $\quad x_n = 1$ $\quad h = \frac{x_n - x_0}{n} = \frac{1-0}{6} = \frac{1}{6}$.

$n = 6$

x	$y = f(x) = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = \frac{1}{1+x_0^2} = \frac{1}{1+0} = 1$
$x_1 = \frac{1}{6}$	$y_1 = \frac{1}{1+x_1^2} = \frac{36}{37} = 0.97297$
$x_2 = \frac{1}{3}$	$y_2 = \frac{1}{1+x_2^2} = \frac{9}{10} = 0.9$
$x_3 = \frac{1}{2}$	$y_3 = \frac{1}{1+x_3^2} = \frac{4}{5} = 0.8$
$x_4 = \frac{2}{3}$	$y_4 = \frac{1}{1+x_4^2} = \frac{9}{13} = 0.6923$
$x_5 = \frac{5}{6}$	$y_5 = \frac{1}{1+x_5^2} = \frac{36}{61} = 0.5902$
$x_6 = 1$	$y_6 = \frac{1}{1+x_6^2} = \frac{1}{1+1} = 0.5$

Trapezoidal Rule.

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{12}\left[(1+0.5) + 2(0.97297+0.9+0.8+0.6923+0.5902)\right]$$

$$= \frac{1}{12}\left[1.5 + 2(3.95547)\right]$$

$$= \frac{1}{12}\left[1.5 + 7.91094\right] = \frac{1}{12}(9.41094)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.784245$$

Simpson's $\frac{1}{3}$ Rule :—

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+y_6+\cdots)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[(1+0.5) + 4(0.97297+0.8+0.5902) + 2(0.9+0.6923)\right]$$

$$= \frac{1}{18}\left[1.5 + 4(2.36317) + 2(1.5923)\right]$$

$$= \frac{1}{18}\left[1.5 + 9.45268 + 3.1846\right] = \frac{1}{18}(14.13728)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.785404444$$

Simpson's $\frac{3}{8}$ Rule :—

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+\cdots)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right].$$

$$= \frac{3}{48}\left[(1+0.5) + 3(0.97297+0.9+0.6923+0.5902) + 2(0.8)\right]$$

$$= \frac{1}{16}\left[1.5 + 9.46641 + 1.6\right] = \frac{1}{16}(12.56641)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.785400625.$$

By direct integration

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx. = \frac{\pi}{4} = 0.78571.$$

Error:- ① (Trapezoidal Rule)

Error = Exact value - obtained value.

$= 0.78571 - 0.784245 = 0.001465.$

② Simpson's $\frac{1}{3}$ Rule.

Error = Exact value - obtained value.

$= 0.78571 - 0.7854044. = 0.0003056.$

③ Simpson's $\frac{3}{8}$ Rule.

Error = Exact value - obtained value.

$= 0.78571 - 0.785400625 = 0.000309375.$

π value due to simpson's $\frac{1}{3}$ Rule. Trapezoidal Rule.

$0.784245 = \frac{\pi}{4} \Rightarrow \pi = 3.13698$

π value due to simpson's $\frac{1}{3}$ Rule.

$0.7854044 = \frac{\pi}{4} \Rightarrow \pi = 3.1416176.$

π value due to simpson's $\frac{3}{8}$ Rule.

$$0.785400625 = \frac{\pi}{4} \Rightarrow \pi = 3.1416025$$

Evaluate $\int_0^{\pi} \sin x \, dx$ by dividing the range into 6 equal parts using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$rd rule (iii) Simpson's $\frac{3}{8}$ rule.

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Sol:- Let $y = f(x) = \sin x$.

Here $a = 0$ $b = \pi$

Number of sub intervals $n = 6$.

$$h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}.$$

$x_0 = 0$, $x_1 = x_0 + h = \frac{\pi}{6}$ $x_2 = x_1 + h = \frac{\pi}{3}$ $x_3 = x_2 + h = \frac{\pi}{2}$.

$x_4 = x_3 + h = \frac{2\pi}{3}$ $x_5 = x_4 + h = \frac{5\pi}{6}$ $x_6 = x_5 + h = \pi$.

$y_0 = \sin x_0 = \sin 0 = 0$. $y_1 = \sin x_1 = \sin\frac{\pi}{6} = 0.5$.

$y_2 = \sin x_2 = \sin\frac{\pi}{3} = 0.866$ $y_3 = \sin x_3 = \sin\frac{\pi}{2} = 1$

$y_4 = \sin x_4 = \sin\frac{2\pi}{3} = 0.866$ $y_5 = \sin x_5 = \sin\frac{5\pi}{6} = 0.5$

$y_6 = \sin x_6 = \sin\pi = 0$.

(i) Trapezoidal rule :-

$$\int_0^{\pi} \sin x \, dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right].$$

$$= \frac{\pi}{12}\left[(0+0) + 2(0.5 + 0.866 + 1 + 0.866 + 0.5)\right]$$

$$= \frac{\pi}{12}[7.464] = \frac{22}{7 \times 12}[7.464] = 1.95486$$

$$\therefore \int_0^{\pi} \sin x \, dx = 1.95486.$$

(ii) Simpson's $\frac{1}{3}$rd rule :-

$$\int_0^{\pi} \sin x \, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right].$$

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$$= \frac{\pi}{18}\left[(0+0)+4(0.5+1+0.5)+2(0.866+0.866)\right]$$

$$= \frac{\pi}{18}\left[8+3.464\right] = \frac{22}{7\times 18}(11.464) = 2.0017.$$

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$$\therefore \int_0^{\pi} \sin x \, dx = 2.0017.$$

(iii) Simpson's $\frac{3}{8}$ rule :-

$$\int_0^{\pi} \sin x \, dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3}{8}\cdot\frac{\pi}{6}\cdot\left[(0+0)+3(0.5+0.866+0.866+0.5)+2(1)\right].$$

$$= \frac{\pi}{16}\left[2+8.196\right] = \frac{22}{16\times 7}(10.196) = 2.0027.$$

$$\therefore \int_0^{\pi} \sin x \, dx = 2.0027.$$

Note :- The volume generated by revolving the area bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$ is given by $\int_a^b \pi y^2 \, dx$.

1) A curve passes through the points (1, 0.2) (2, 0.7) (3, 1) (4, 1.3) (5, 1.5) (6, 1.7) (7, 1.9) (8, 2.1) (9, 2.3). Using (i) Trapezoidal rule (ii) Simpson's $\frac{1}{3}$rd rule estimate the volume generated by revolving the area between the curve, the x-axis and the ordinates $x=1$ and $x=9$ about the x-axis.

Sol:- The volume generated by revolving the area bounded by the curve $y=f(x)$ the x-axis and the ordinates $x=a$ and $x=b$ is $\int_{x=a}^{x=b} \pi y^2 \, dx$.

Here $a=1$, $b=9$. $h=\frac{b-a}{n} = \frac{9-1}{8} = 1$

$y_0 = 0.2 \quad y_0^2 = 0.04 \qquad y_1 = 0.7 \quad y_1^2 = 0.49$

$y_2 = 1 \quad y_2^2 = 1 \qquad y_3 = 1.3 \quad y_3^2 = 1.69$

$y_4 = 1.5 \quad y_4^2 = 2.25 \qquad y_5 = 1.7 \quad y_5^2 = 2.89$

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$y_6 = 1.9 \quad y_6^2 = 3.61 \qquad y_7 = 2.1 \quad y_7^2 = 4.41$

$y_8 = 2.3 \quad y_8^2 = 5.29.$

(i) Trapezoidal rule :-

Required volume of the solid $= \pi \int_{x=1}^{x=9} y^2\, dx.$

$$= \pi \frac{h}{2}\left[(y_0^2 + y_8^2) + 2(y_1^2 + y_2^2 + \cdots + y_7^2)\right]$$

$$= \frac{\pi}{2}\left[(0.04 + 5.29) + 2(0.49 + 1 + 1.69 + 2.25 + 2.89 + 3.61 + 4.4)\right]$$

$$= \frac{22}{7 \times 2}\left[5.33 + 32.66\right] = \frac{22}{14}\left[37.99\right] = 59.699.$$

$\therefore$ Volume of the solid $= 59.699$ cubic units.

(ii) Simpson's $\frac{1}{3}$rd rule :—

Required volume of the solid $= \pi \int_{x=1}^{x=9} y^2\, dx.$

$$= \pi \cdot \frac{h}{3}\left[(y_0^2 + y_8^2) + 4(y_1^2 + y_3^2 + y_5^2 + y_7^2) + 2(y_2^2 + y_4^2 + y_6^2)\right]$$

$$= \pi \cdot \frac{1}{3}\left[(0.04 + 5.29) + 4(0.49 + 1.69 + 2.89 + 4.41) + 2(1 + 2.25 + 3.61)\right].$$

$$= \frac{22}{21}\left[5.33 + 37.92 + 13.72\right] = 59.68.$$

$\therefore$ volume of the solid $= 59.68$ cubic units.

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The velocity v of a particle at a distance s from a point on its path. (17)

is given by the following table.

s (ft)	0	10	20	30	40	50	60
v (ft/s)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft using (i) Simpson's $\frac{1}{3}$rd rule (ii) Simpson's $\frac{3}{8}$th rule. (iii) Trapezoidal rule.

Sol:- We know that the velocity is the rate of change of displacement.

$$\text{i.e } \frac{ds}{dt} = v.$$

The time taken to travel 60 ft is given by

$$dt = \frac{ds}{v}$$

$$t = \int dt = \int_0^{60} \frac{1}{v}\, ds$$

$$t = \int_0^{60} \frac{1}{v}\, ds$$

Take $s = x$ and $y = \frac{1}{v}$.

$$t = \int_0^{60} y\, dx$$

$y_0 = \frac{1}{v_0} = \frac{1}{47} = 0.02128$ $\quad y_1 = \frac{1}{v_1} = \frac{1}{58} = 0.01724$

$y_2 = \frac{1}{v_2} = \frac{1}{64} = 0.015625$ $\quad y_3 = \frac{1}{v_3} = \frac{1}{65} = 0.01538$

$y_4 = \frac{1}{v_4} = \frac{1}{61} = 0.016393$ $\quad y_5 = \frac{1}{v_5} = \frac{1}{52} = 0.019231$

$y_6 = \frac{1}{v_6} = \frac{1}{38} = 0.026316.$ $\quad$ Here $h = 10$. $\left[\because h = \frac{b-a}{n}\right.$

$$\left. h = \frac{60-0}{6} = 10\right]$$

Simpson's $\frac{1}{3}$rd rule :-

$$t = \int_0^{60} y\, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{10}{3}\left[(0.02128 + 0.026316) + 4(0.01724 + 0.01538 + 0.019231) + 2(0.015625 + 0.016393)\right].$$

$$= \frac{10}{3}\left[0.047596 + 0.207404 + 0.064036\right]$$

$$t = 1.06345$$

(18)

(ii) Simpson's $\frac{3}{8}$th rule :—

$$t = \int_0^{60} y\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3(10)}{8}\left[(0.02128 + 0.026316) + 3(0.01724 + 0.015625 + 0.016393 + 0.019231) + 2(0.01538)\right]$$

$$= \frac{30}{8}\left[0.047596 + 0.205467 + 0.03076\right]$$

$$t = 1.0643.$$

(1) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule by taking $h=0.5$ $h=0.25$ and $h=0.125$.   Ans: 0.775, 0.7828, 0.78415.

(2) A rocket is launched from the ground. Its acceleration measured every 5 seconds is tabulated below. Find the velocity and position of the rocket at $t = 40$ seconds. Use Trapezoidal rule as well as Simpson's rule.

t	0	5	10	15	20	25	30	35	40
a	40	45.25	48.5	51.25	54.35	59.48	61.5	64.3	68.7

Ans: Trapezoidal rule — 2194.9, 87796.

Simpson's rule — 2197.5, 87900.

(3) Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range of integration into 4 equal parts using (a) Trapezoidal rule (b) Simpson's $\frac{1}{3}$rd rule.

Ans:- 0.7428, 0.7467

(4) Find the area under the curve represented by the following table bounded by x-axis and the ordinates 0.6 and 1.2

x	0.6	0.8	1.0	1.2
y	1.23	1.58	2.03	4.32.

Ans: 1.277.

(5) Evaluate $\int_0^{\pi} \sin x \, dx$ by dividing the range into 10 equal parts using (a) Trapezoidal rule (b) Simpson's $\frac{1}{3}$rd rule. Ans:- 1.9835, 2.0007.

(6) Evaluate $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$ taking $h = \frac{\pi}{12}$ using (a) Trapezoidal rule. (b) Simpson's $\frac{1}{3}$rd rule   Ans:- 1.1702, 1.18718.

(7) Find $\int_0^1 \frac{dx}{1+x}$ using 10 intervals using simpson's rule

Ans:- 0.6931684.

(8) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by taking $h = \frac{1}{6}$ using (i) simpson's $\frac{1}{3}$rd rule.

$h = \frac{1}{6}$ using (ii) simpson's $\frac{3}{8}$th rule.

Ans:- 0.785388, 0.785398

(9) The velocities of a car at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
Velocity in km/hr	0	22	30	27	18	7	0

Find the distance covered by the car.

(i) Using simpson's $\frac{1}{3}$rd rule (ii) Using simpson's $\frac{3}{8}$th rule.

Ans:- 3.5555, 3.5625.

(10) The velocity v of a particle at a distance s from a point on its path is given by the following table.

s (ft)	0	10	20	30	40	50	60
v (ft/s)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft using.

(i) simpson's $\frac{1}{3}$rd rule (ii) simpson's $\frac{3}{8}$th rule.

Ans:- 1.063518, 1.0643723.

(11) A curve passes through the points (1, 0.2) (2, 0.7) (3, 1) (4, 1.3) (5, 1.5) (6, 1.7) (7, 1.9) (8, 2.1) (9, 2.3) using simpson's $\frac{1}{3}$rd rule estimate the volume generated by revolving the area between the curve the x-axis and the ordinates $x=1$ and $x=9$ about the x-axis.

Ans:- 18.99.

(12) A curve is drawn to pass through the points given by the following table.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Find the area bounded by the curve the x-axis and the ordinates $x=1$ and $x=4$. Ans :- 7.74. (3)

(13) A solid of revolution is formed by rotating about the x-axis the area b/n x-axis and times $x=0$ and $x=1$ and a curve through the points with the following co ordinate.

x	0	0.25	0.5	0.75	1
y	1	0.9896	0.9589	0.9059	0.8415

Find the volume of the solid formed Ans:- 2.8192

(14) The velocity v of a particle at distance s from a point in its path is given by

s	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60ft using simpson's $\frac{1}{3}$rd rule.

Ans:- 1.063.

(15) calculate $\int_0^{\pi/2} e^{\sin x}\,dx$ using simpson's rule by taking 7 ordinates and correct to 4 decimal places Ans:- 3.1043.

(16) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using simpson's $\frac{1}{3}$rd, simpson's $\frac{3}{8}$th and Trapezoidal rule - Hence obtain the approximate value of π in each. Compare the result with exact value.

(17) Find $\int_0^1 \frac{x^2}{1+x^3} dx$. using simpson's $\frac{1}{3}$rd rule by taking 4 Subintervals.

Also find the error. Ans:- 0.231066, 0.000026. (4)

(18) The velocity of a train which starts from rest is given by the following table.

t min	2	4	6	8	10	12	14	16	18	20
V km/hr	16	28.8	40	46.4	51.2	32	17.6	8	3.2	0

Estimate total distance run in 20 min. Ans:- 8.25 km.

(19) Find the value of $\int_1^5 \log_{10} x \, dx$ taking 8 sub intervals correct to 4 significant figures by Trapezoidal and simpson's $\frac{1}{3}$rd rule.

Ans. 1.7505025.

(20) The table below shows the velocities of a moped which starts from rest at fixed intervals of time. Find the distance traveled by the moped in 20 min.

Time (t)	2	4	6	8	10	12	14	16	18	20
Velocity (v)	0	10	18	25	29	32	20	11	5	2

Ans:- 309.33 km.

(21) Evaluate $\int_{0.2}^{1.4} (\sin x - \log_e^x + e^x) dx$ by dividing the interval into (i) 6 equal parts

(ii) 12 equal parts Ans:- 4.05116.

(22) A solid of revolution is formed by rotating about the x-axis, the area between the x-axis the lines $x=0$ and $x=1$ and a curve through the points with the following co ordinates.

x	0	0.25	0.5	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using simpson's rule.
Ans:- 2.8192.

23) The velocity v of a particle at distance s from a point on its linear path is given by the following table.

5

s (m)	0	2.5	5	7.5	10	12.5	15	17.5	20
v (m/sec)	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 metres using simpson's rule.

24) The velocity v of a particle at distances s from a point on its path given by the table.

s ft	0	10	20	30	40	50	60
v ft/sec	47	58	64	65	61	52	38.

Estimate the time taken to travel 60 ft by using simpson's $\frac{1}{3}$ rule. Compare the result with simpson's $\frac{3}{8}$ rule.

25) The following table gives the velocity v of a particle at time t.

t (sec)	0	2	4	6	8	10	12
v (m/sec)	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds and also the acceleration at t = 2 sec.

26) A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at t = 80 sec.

t (sec)	0	10	20	30	40	50	60	70	80
f (cm/sec²)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

27) A curve is drawn to pass through the points given by the following table.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve, x-axis and the lines x = 1, x = 4

28) A river is 80 ft wide. The depth d in feet at a distance x ft from one bank is given by the following table.

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3.

Find approximately the area of the cross section

A rocket is launched from the ground Its accelerat

29) A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below.

h (ft)	10	11	12	13	14
A (sq.ft)	950	1070	1200	1350	1530.

If t denotes time in minutes, the rate of fall of the surface is given by $\frac{dh}{dt} = -48 \frac{\sqrt{h}}{A}$ Estimate the time taken for the water level to fall from 14 to 10 ft above the sluices.

30) Evaluate (a) $\int_1^2 \cos x \, dx$ using the trapezoidal rule with $h = \frac{1}{2}$. Compare with the exact solution.

(b) $\int_0^{\pi/2} e^{-x} \cos x \, dx$ using the simpson's $\frac{1}{3}$rd rule with $h = \frac{\pi}{8}$.

R.NO	Q.NO
1-20	1-10
21-40	11-20
41-60	21-30